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SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF SOME INFINITE LINEAR CHAINS AND TWO-DIMENSIONAL ARRAYS

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To Eloise—*sine qua non.*

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CHAPTER I

INTRODUCTION

Various phenomena of the natural sciences may be described by mathematical models which consist of countably infinite systems of ordinary differential equations with constant coefficients and some set of specified initial conditions [5,6,8,9,14,16,17,19,20,21,24,25]. The purpose of this dissertation is to investigate several such mathematical models and to describe a procedure for constructing solutions of the denumerably infinite initial-value problems.

Associated with any problem in differential equations are the questions of existence and uniqueness of solutions, properties of solutions, and the explicit representation of solutions when possible. Theorems on existence, uniqueness, and general properties of solutions of denumerable systems of differential equations are known in many cases [10,18,22,26], and such matters are not the primary objective of this investigation. The major goal here is the explicit representation of solutions--that is, the construction of a denumerable sequence of sufficiently differentiable functions which reduce each of the differential equations to an identity and satisfy the prescribed initial conditions. Clearly whenever this goal is achieved, the question of existence is answered in the affirmative, and many properties of the solution can be deduced directly from the explicit representation.

The systems of differential equations considered are associated with both one- and two-dimensional physical models--for example, infinite chains of coupled linear oscillators or planar arrays of frictionally coupled masses located at lattice points. For the one-dimensional problems the differential equations are of the form $\dot{x} = Ax$ or $\ddot{x} = Ax + B\dot{x}$, where x is an infinite column vector, A is an infinite tridiagonal matrix of real constants, and B is an infinite scalar matrix. The elements of the infinite matrix A are used to construct a three-term recurrence relation which generates a sequence of orthogonal polynomials. The recurrence property of these polynomials is used in a separation-of-variables technique in which the only differential equation to be solved is a scalar ordinary differential equation containing the argument of the polynomials as a parameter. The orthogonality property is utilized to satisfy the prescribed initial conditions by superposing continuously over the interval of orthogonality. Thus an integral representation is obtained for individual component functions x_n of the solution $x = \{x_n\}$ of the denumerable one-dimensional initial-value problem. The two-dimensional problems considered are essentially Cartesian products of two identifiable one-dimensional problems. The coefficients in the differential equations of the corresponding one-dimensional problems are used to construct two three-term recurrences each of which determines a sequence of orthogonal polynomials of one variable--say $\{P_n(x)\}$ and $\{Q_n(y)\}$. The biorthogonal sequence*

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*A sequence of functions $\{f_{ij}\}$ defined on a region R of the plane is a *biorthogonal sequence* with respect to the inner product $\langle\, ,\, \rangle$ if $\langle f_{ij}, f_{pq} \rangle \neq 0$ only if i=p and j=q.

$\{f_{ij}(x,y)\}$ (where $f_{ij}(x,y) = P_i(x)Q_j(y)$) is used to obtain a double integral representation of the component functions x_{ij} of the solution of the two-dimensional denumerable initial-value problem.

As might be inferred from the preceding paragraph, the solution of the problems considered depends on the use of polynomials generated by a recurrence of the form $P_{n+1}(x) = (A_n x+B_n)P_n(x) - C_n P_{n-1}(x)$ $(n\geq 0)$ with $P_{-1}(x) \equiv 0$ and $P_0(x) = 1$. Jayne [12] has determined necessary and sufficient conditions on the recurrence coefficients A_n $(n\geq 0)$, B_n $(n\geq 0)$, and $C_n(n\geq 1)$ for the recursively generated polynomials to be Sturm-Liouville polynomial sequences associated with a second-order differential equation. Whenever these conditions are satisfied, the determination of the interval of orthogonality and the weight function is easy. The condition $\dfrac{C_n}{A_n A_{n-1}} > 0$ $(n\geq 1)$ has been shown by Favard [7] and Law [15] to be both a necessary and sufficient condition on the recurrence coefficients for the recursively generated polynomials to be orthogonal on *some* interval I of the real line with respect to *some* integrator α (a bounded, real-valued, non-decreasing function which assumes infinitely many different values on I). However, when Jayne's conditions are not satisfied, determining the interval and integrator may be difficult. For several such cases, the interval and the integrator are displayed in Chapter IV, although a practicable general method of determining them in terms of the recurrence coefficients cannot yet be given.

The discussion in Chapter I is primarily for orientation. Chapter II deals with the treatment of one-dimensional initial-value problems. In preparation for that work, those aspects of the theory of orthogonal polynomials needed in the remainder of the chapter are introduced. The first initial-value problem considered is an extension of the problems treated by Law [15] and has the form $\ddot{x} = Ax + B\dot{x}$, where $x = \{x_n\}_{n=0}^{\infty}$, A is a tridiagonal matrix, and B is a scalar matrix (in Law's work the first-derivative terms are absent). Due largely to the appearance of corresponding physical systems, this problem is referred to as a half-infinite initial-value problem; and its solution is given in Theorem 2. The remainder of Chapter II is devoted to the solution of a class of pth-order infinite initial-value problems of the form $x^{(p)} = Ax + B_1\dot{x} + B_2\ddot{x} + \ldots + B_{p-1}x^{(p-1)}$, where $x = \{x_n\}_{-\infty}^{\infty}$, A is tridiagonal matrix with real elements that satisfy the symmetry requirement $a_{ij} = a_{-i,-j}$ $(i \geq 1,\ j \geq 1)$, and $B_1, B_2, \ldots, B_{p-1}$ are infinite scalar matrices. The symmetry restrictions imposed on A are equivalent to physical symmetry (about a point designated the middle) of the corresponding physical systems. Solutions of the infinite problem are obtained by decomposing it into two half-infinite problems.

A considerable portion of Chapter III is devoted to a description of two-dimensional problems and some corresponding physical systems. Finite truncations obtained from the infinite systems by simply deleting all but a finite number of the differential equations of the infinite systems are also introduced. It is shown that the exact solutions of these finite systems are obtained by a suitable quadrature approximation of the double-integral representations of the solutions of the corresponding infinite systems. Error estimates are furnished for the difference between any component of the solution of a finite truncation and its counterpart in the solution of the infinite system. It is of some interest that the solution of a finite system requires the a priori knowledge of the zeros of the associated polynomials, whereas the solution of the infinite system does not require that the zeros be known. Thus the solution of the infinite system is an attractive approximation to the more cumbersome solution of a large finite system.

Chapter IV contains a detailed analysis of two sequences of nonclassical orthogonal polynomials, $\{M_n^{(\alpha,\beta)}\}$ and $\{A_n^{(\alpha)}\}$, each of which is generated by a three-term recurrence relation in which the recurrence coefficients depend on the parameters α and β. Representations are given for $M_n^{(\alpha,\beta)}$ and $A_n^{(\alpha)}$ for all values of $\alpha>0$, $\beta>0$. Possibly the most interesting part of Chapter IV is the dependence of the interval of orthogonality and weight function (or integrator) on the parameters α and β. It is shown that for $\alpha=1$ and $\beta>1$ the polynomials $\{M_n^{(1,\beta)}\}$ are orthogonal on $[0,4]$ with respect to a weight function; but if $\alpha=1$ and $\beta<1$, the polynomials $\{M_n^{(1,\beta)}\}$ are orthogonal on $\left[0, \frac{4}{\beta(2-\beta)}\right]$ with respect to an integrator with a jump at $\frac{4}{\beta(2-\beta)}$. The transition from the existence of a weight function to the requirement of an integrator occurs at $\alpha=1$, $\beta=1$. With the aid of [11] it is shown that the polynomials $\{M_n^{(1,1)}\}$ are classical polynomials. Similar results are shown for $\beta = 2$: for $\alpha<1$, $\{M_n^{(\alpha,2)}\}$ are orthogonal on $[0,2] \cup \left[\frac{2}{\alpha}, 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ with respect to a weight function; for $\alpha>1$, $\{M_n^{(\alpha,2)}\}$ are orthogonal on $\left[0, 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ with respect to an integrator with a jump at $\left(\frac{\alpha+1}{\alpha}\right)$; and in the transitional case where $\alpha=1$, the $\{M_n^{(1,2)}\}$ are again classical polynomials. The polynomials $\{A_n^{(\alpha)}\}$ are shown always to be orthogonal with respect to a weight function.

Three examples which are presented in Chapter V are mathematical models associated with infinite linear chains with isotopic impurities. The non-classical polynomials described in Chapter IV are used in conjunction with the results of Chapter II to construct solutions of these models. An interesting relationship between the frequency spectrum of the physical system and the qualitative properties of the integrator is pointed out.

Appendix A contains a development of the quadrature formula for double integrals used in Chapter III, and Appendix B contains a catalogue of several physical systems with their corresponding solutions.

CHAPTER II

SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF SOME INFINITE LINEAR CHAINS

This chapter contains three major sections. The first is devoted to a summary of some ideas about orthogonal polynomials which are used in solving the initial-value problems considered in the remainder of this chapter and in Chapter III. In the second section a procedure is described for constructing a solution of the half-infinite initial-value problem

$$\ddot{x} = Ax + B\dot{x},$$

$$x(0) = x_0,\ \dot{x}(0) = \dot{x}_0,$$

where $A = (a_{ij})$, $i\geq 1$, $j\geq 1$, is an infinite tridiagonal matrix of real constants and $B = (\beta\delta_{ij})$, $i\geq 1$, $j\geq 1$, is an infinite scalar matrix of real constants. The half-infinite initial-value problem $\ddot{x} = Ax + B\dot{x}$, $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ may be construed as a mathematical model for several conceivable physical systems. One such system (see Figure 1) is a half-infinite chain of coupled linear oscillators in which each mass is subjected to viscous damping proportional to its mass. The final section of the chapter details the construction of a solution of a class of infinite initial-value problems in which the operator in each of the equations is of pth order and the infinite coefficient matrix has certain symmetry properties to be described.

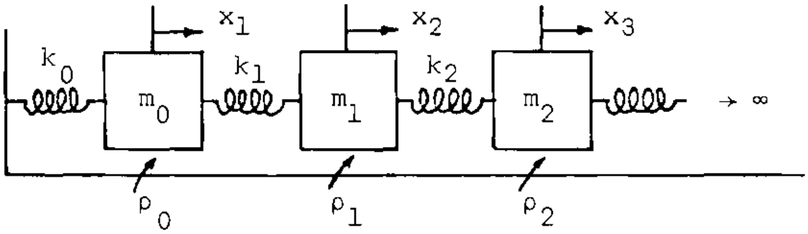

Figure 1. A Half-Infinite System of Damped Oscillators with $\frac{\rho_n}{m_n} = \beta$, $n \geq 0$

Two problems of this class receive special attention because of their importance as mathematical models of physical phenomena. In particular, in Theorems 4 and 5 solutions are given for the systems

$$\dot{v} = Cv,$$

$$v(0) = v_0,$$

and

$$\ddot{x} = Cx + D\dot{x},$$

$$x(0) = x_0,\ \dot{x}(0) = \dot{x}_0,$$

where $C = (c_{ij})$, $-\infty<i,j<\infty$, is an infinite tridiagonal matrix of real constants with $c_{ij} = c_{-i,-j}$ $(i\geq 1,\ j\geq 1)$ and $D = (\beta\delta_{ij})$, $-\infty<i,j<\infty$, is a real scalar matrix. The infinite initial-value problem $\dot{v} = Cv$,

$v(0) = v_0$ may be viewed as a mathematical model of an infinite stack of sliding plates extending indefinitely above and below some point of physical symmetry in the stack (see Figure 2).

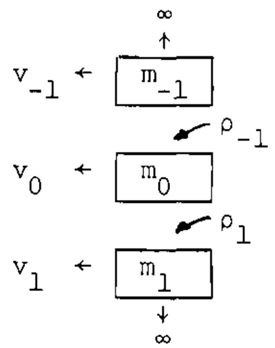

Figure 2. A Physically Symmetric Infinite Stack of Sliding Plates

The infinite initial-value problem $\ddot{x} = Cx + D\dot{x}$, $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ may be viewed as a mathematical model of an infinite chain of coupled linear oscillators extending indefinitely to the left and right from some point of physical symmetry in the chain, each mass subjected to viscous damping proportional to its mass (see Figure 3).

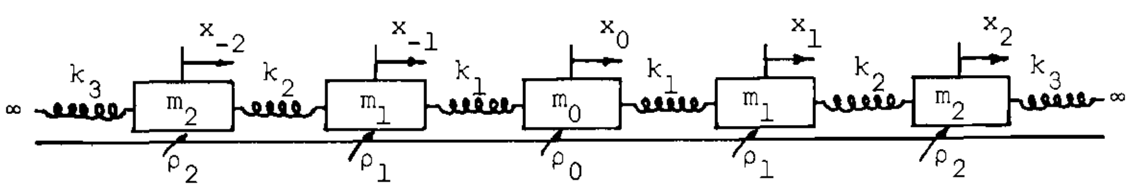

Figure 3. A Physically Symmetric Infinite System of Damped Oscillators with $\rho_n/m_n = \beta$, $n \geq 0$

Recursively Generated Polynomials

A three-term recurrence

$$P_0 = 1 \tag{1}$$

$$P_1(x) = A_0x + B_0$$

$$P_{n+1}(x) = (A_nx+B_n)P_n(x) - C_nP_{n-1}(x), \quad n\geq 1,$$

where A_n $(n\geq 0)$, B_n $(n\geq 0)$, C_n $(n\geq 1)$ are real constants and $A_n \neq 0$ $(n\geq 0)$, generates a sequence of polynomials $\{P_n\}$ in which P_n is of degree exactly n. Some of the properties of such recursively generated polynomial sequences are stated here for convenient reference.

Definition 1. An *integrator* α is a bounded, non-decreasing, real-valued function which is defined on an interval [a,b] of the real line and assumes infinitely many different values on [a,b].

Definition 2. A sequence of polynomials $\{P_n\}$ is said to be *orthogonal on [a,b] with respect to an integrator* α if

$$\int_a^b P_i(x)P_j(x)d\alpha(x) = 0, \quad i\neq j,$$

where the integral is a Stieltjes integral.

The following theorem [7,15] is of fundamental importance in deciding whether the polynomials generated by (1) are orthogonal with respect to some integrator α on some interval $[a,b]$ of the real line.

Theorem 1. The polynomials generated by (1) are orthogonal on some interval $[a,b]$ of the real line with respect to some integrator α if and only if the recurrence coefficients satisfy the condition

$$\frac{C_n}{A_n A_{n-1}} > 0 \quad (n \geq 1). \tag{2}$$

It is easily seen that if $\{P_n\}$ is a sequence of polynomials and α is an integrator such that $\int_a^b P_i(x)P_j(x)d\alpha(x) = 0$, $i \neq j$, then $\int_a^b P_i(x)P_j(x)d[c_1\alpha(x)+c_2] = 0$, $i \neq j$, for any real constants c_1, c_2. For the sake of standardization the following convention is adopted in the sequel.

Definition 3. A *normalized integrator* α is an integrator which satisfies the conditions $\int_a^b d\alpha(x) = 1$ and $\alpha(a) = 0$.

For many sequences of polynomials which are generated by (1) and for which (2) holds, the corresponding integrator α is absolutely continuous on (a,b) and $d\alpha(x) = \rho(x)dx$.

Definition 4. A *normalized weight function* ρ is a non-negative real-valued function defined on an interval (a,b) of the real line, integrable on $[a,b]$ (at least improperly), and such that $\int_a^b \rho(x)dx = 1$.

Definition 5. A sequence of polynomials $\{P_n\}$ is said to be *orthogonal on [a,b] with respect to the normalized weight function* ρ if the Riemann integral $\int_a^b P_i(x)P_j(x)\rho(x)dx = 0,\ i\neq j$.

If a sequence $\{P_n\}$ of polynomials is generated by (1) and if the condition (2) holds, the square of the $L_2(\alpha)$ norm of the polynomial P_n may be calculated directly from the recurrence coefficients [15].

Lemma 1. If $\{P_n\}$ is a sequence of polynomials generated by (1) and $\frac{C_n}{A_nA_{n-1}} > 0$ for $n\geq 1$, then

$$\gamma_n \overset{\Delta}{=} \int_a^b P_n^2(x)d\alpha(x) = \frac{A_0}{A_n} C_0C_1\ldots C_n \quad \text{for } n\geq 0,$$

where $C_0 \overset{\Delta}{=} 1$ for convenience.

Proof. For each integer $n\geq 0$,

$$\gamma_{n+1} = \int_a^b P_{n+1}^2(x)d\alpha(x) = \int_a^b P_{n+1}(x)[A_nxP_n(x)+B_nP_n(x)-C_nP_{n-1}(x)]d\alpha(x)$$

$$= A_n \int_a^b xP_{n+1}(x)P_n(x)d\alpha(x).$$

From (1),

$$xP_{n+1}(x) = \frac{P_{n+2}(x)}{A_{n+1}} - \frac{B_{n+1}}{A_{n+1}} P_{n+1}(x) + \frac{C_{n+1}}{A_{n+1}} P_n(x).$$

Thus

$$\gamma_{n+1} = A_n \int_a^b \left[\frac{P_{n+2}(x)}{A_{n+1}} - \frac{B_{n+1}}{A_{n+1}} P_{n+1}(x) + \frac{C_{n+1}}{A_{n+1}} P_n(x)\right] P_n(x)d\alpha(x)$$

$$= \frac{A_n C_{n+1}}{A_{n+1}} \int_a^b P_n^2(x)d\alpha(x) = \frac{A_n C_{n+1}}{A_{n+1}} \gamma_n .$$

From the relation $\gamma_{n+1} = \dfrac{A_n C_{n+1}}{A_{n+1}} \gamma_n$ $(n\geq 0)$ and $C_0 = \gamma_0 = 1$, a straightforward induction proof yields the result.

Solution of the Half-Infinite System $\ddot{x} = Ax + B\dot{x}$

In this section a procedure is described for constructing a sequence of functions which provide a solution of the half-infinite initial-value problem consisting of the differential equations

$$A_0\ddot{x}_0 + \beta A_0 x_0 - B_0\dot{x}_0 + x_1 = 0 \tag{3}$$

$$C_n x_{n-1} + A_n\ddot{x}_n + \beta A_n\dot{x}_n - B_n x_n + x_{n+1} = 0 \quad (n\geq 1)$$

together with the initial conditions

$$x_k(0) = a_k,\ \dot{x}_k(0) = b_k \tag{3.1}$$

$$x_n(0) = 0,\ \dot{x}_n(0) = 0,\ n\neq k,\ n\geq 0.$$

Here $\cdot \sim \dfrac{d}{dt}$; $A_n \neq 0$ $(n\geq 0)$; $\dfrac{C_n}{A_n A_{n-1}} > 0$ $(n\geq 1)$; and k is a fixed non-negative integer.

The method originally used to solve this problem, while correct, was inelegant. A separation technique used by S. Karlin and J. L. McGregor [14], and extended to more general systems by W. G. Christian [4] during a current investigation of countably infinite systems, yields a solution in a more skillful way. This technique is employed in what follows.

Associated with the differential equations (3), consider the three-term recurrence relation

$$P_0 = 1$$

$$P_1(x) = A_0 x + B_0$$

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \ n \geq 1.$$

Lemma 2. The differential equations (3) have a solution $\{x_n(t)\}$ of the form

$$x_n(t) = P_n(x)u(x,t) \qquad (n \geq 0) \tag{4}$$

if and only if

$$\frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} + xu = 0 \tag{5}$$

for each x.

Proof. If (4) is substituted into the differential equations (3) and the recurrence (1) is used to replace P_{n+1}, the (n+1)th equation of (3) becomes

$$A_n P_n(x)\left[\frac{\partial^2 u}{\partial t^2} + \beta\,\frac{\partial u}{\partial t} + xu\right] = 0, \quad n\geq 0. \tag{6}$$

Since $A_0 \neq 0$, (6) clearly holds for n=0 if and only if (5) is satisfied. For $n\geq 1$, $A_n \neq 0$ and $\frac{C_n}{A_n A_{n-1}} > 0$; so the polynomials $\{P_n\}$ are orthogonal on some interval [a,b] with respect to some integrator α. Thus there exists no x such that $P_n(x) = 0$ for all $n\geq 1$.* It follows that (6) holds for all $n\geq 0$ and each x if and only if u(x,t) satisfies (5).

Lemma 3. The general solution of (5) is

$$u(x,t) = \begin{cases} e^{-\frac{\beta t}{2}}\left\{c_1(x,k)\cos\sqrt{x - \frac{\beta^2}{4}}\,t + c_2(x,k)\,\dfrac{\sin\sqrt{x - \frac{\beta^2}{4}}\,t}{\sqrt{x - \frac{\beta^2}{4}}}\right\}, & x \geq \frac{\beta^2}{4}, \\[2ex] e^{-\frac{\beta t}{2}}\left\{c_1(x,k)\cosh\sqrt{\frac{\beta^2}{4} - x}\,t + c_2(x,k)\,\dfrac{\sinh\sqrt{\frac{\beta^2}{4} - x}\,t}{\sqrt{\frac{\beta^2}{4} - x}}\right\} & x < \frac{\beta^2}{4}, \end{cases} \tag{7}$$

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*It is easily shown that the zeros of the nth polynomial $P_n(x)$ are real and simple and that they all lie in the open interval (a,b). Furthermore the zeros of $P_{n+1}(x)$ interlace the zeros of $P_n(x)$, and no zero of $P_n(x)$ is a zero of $P_{n+1}(x)$.

where $c_1(x,k)$, $c_2(x,k)$ are arbitrary functions of x,k but independent of n,t.

Proof. This conclusion follows immediately from (5) with x treated as a parameter.

The results of Lemmas 2 and 3 show that $x_n(t) = P_n(x)u(x,t)$ provide a solution of the system of differential equations (3) for any choice of $c_1(x,k)$ and $c_2(x,k)$ in (7). Thus to obtain a solution of the initial-value problem (3)-(3.1), it is only necessary to choose $c_1(x,k)$ and $c_2(x,k)$ so as to satisfy the initial conditions (3.1). It is in this choice of $c_1(x,k)$ and $c_2(x,k)$ that the orthogonality of the polynomials is used.

Theorem 2. Suppose that, in system (3), $\frac{C_n}{A_n A_{n-1}} > 0$ for $n=1,2,3,\ldots$. Let the sequence of polynomials $\{P_n\}$, generated by (1), be orthogonal on the interval $[a,b]$ with respect to the normalized integrator α. For each non-negative integer n let

$$\gamma_n = \int_a^b P_n^2(x)d\alpha(x) = \frac{A_0}{A_n} C_0 C_1 \ldots C_n;$$

and define

$$X_n(t) = \int_a^b P_n(x)K(x,t)d\alpha(x), \tag{8}$$

where

$$K(x,t) = \begin{cases} \dfrac{e^{-\frac{\beta t}{2}} P_k(x)}{\gamma_k}\left[a_k \cos\sqrt{x - \frac{\beta^2}{4}}\, t + \left(b_k + \dfrac{\beta a_k}{2}\right)\dfrac{\sin\sqrt{x - \frac{\beta^2}{4}}\, t}{\sqrt{x - \frac{\beta^2}{4}}}\right], & x \geq \dfrac{\beta^2}{4} \\[2ex] \dfrac{e^{-\frac{\beta t}{2}} P_k(x)}{\gamma_k}\left[a_k \cosh\sqrt{\frac{\beta^2}{4} - x}\, t + \left(b_k + \dfrac{\beta a_k}{2}\right)\dfrac{\sinh\sqrt{\frac{\beta^2}{4} - x}\, t}{\sqrt{\frac{\beta^2}{4} - x}}\right], & x < \dfrac{\beta^2}{4}. \end{cases}$$

Suppose that for each $t \geq 0$,

$$\dot{x}_n(t) = \int_a^b P_n(x)\, \frac{\partial K}{\partial t}(x,t)\, d\alpha(x),\quad \ddot{x}_n(t) = \int_a^b P_n(x)\, \frac{\partial^2 K}{\partial t^2}(x,t)\, d\alpha(x) \qquad (9)$$

(that is, differentiation under the integral sign is permissible). Then the sequence $\{x_n\}$ defined by (8) is a solution of the half-infinite initial-value problem (3)-(3.1).

Proof. That $x_n(t)$, as defined by (8), satisfies the differential equations (3) follows immediately from Lemmas 2 and 3. From the orthogonality of the polynomials P_n and the first of hypotheses (9) it is easily seen that

$$x_n(0) = \int_a^b P_n(x)\, \frac{P_k(x)}{\gamma_k}\, a_k\, d\alpha(x) = a_k \delta_{nk},$$

$$\dot{x}_n(0) = \int_a^b P_n(x)\, \frac{P_k(x)}{\gamma_k}\left[-\frac{\beta}{2} a_k + \left(b_k + \frac{\beta a_k}{2}\right)\right] d\alpha(x) = b_k \delta_{nk},$$

and the initial conditions (3.1) are satisfied. This completes the proof.

Solutions of Infinite Initial-Value Problems

A seemingly natural extension of the half-infinite initial-value problems considered in the previous section and in [15] is the infinite initial-value problem. First, two such problems are described; then they are framed in a more general setting, and a procedure is given for constructing solutions.

Let k be a prescribed non-negative integer, and $\{A_n\}$, $-\infty<n<\infty$, $\{B_n\}$, $-\infty<n<\infty$, $\{C_n\}$, $-\infty<n<\infty$, sequences of real numbers with the properties

$$\begin{gathered} A_n = A_{-n}\ (n\geq 0),\ A_n \neq 0\ (n\geq 0), \\ B_n = B_{-n}\ (n\geq 0), \\ C_n = C_{-n}\ (n\geq 0),\ C_n > 0\ (n\geq 1),\ C_0 = 1, \\ \frac{C_n}{A_n A_{n-1}} > 0\ (n\geq 1). \end{gathered} \tag{10}$$

Now consider the first-order infinite initial-value problem

$$\begin{gathered} x_{-n-1} + A_{-n}\dot{x}_{-n} - B_{-n}x_{-n} + C_{-n}x_{-n+1} = 0\ (n\geq 1) \\ x_{-1} + A_0\dot{x}_0 - B_0x_0 + x_1 = 0 \end{gathered} \tag{11}$$

$$C_n x_{n-1} + A_n \dot{x}_n - B_n x_n + x_{n+1} = 0 \quad (n \geq 1)$$

with the initial conditions

$$x_k(0) = a_k \neq 0$$
$$x_n(0) = 0, \; n \neq k, \; -\infty < n < \infty, \tag{11.1}$$

and the second-order infinite initial-value problem

$$x_{-n-1} + A_{-n}\ddot{x}_{-n} + A_{-n}\beta\dot{x}_{-n} - B_{-n}x_{-n} + C_{-n}x_{-n+1} = 0 \quad (n \geq 1)$$

$$x_{-1} + A_0\ddot{x}_0 + A_0\beta\dot{x}_0 - B_0 x_0 + x_1 = 0 \tag{12}$$

$$C_n x_{n-1} + A_n\ddot{x}_n + A_n\beta\dot{x}_n - B_n x_n + x_{n+1} = 0 \quad (n \geq 1)$$

(where β is a real constant) with the initial conditions

$$x_k(0) = a_k, \; \dot{x}_k(0) = b_k$$
$$x_n(0) = 0, \; \dot{x}_n(0) = 0, \; n \neq k, \; -\infty < n < \infty. \tag{12.1}$$

The infinite initial-value problems (11)-(11.1) and (12)-(12.1) will be solved as special cases of the procedure developed below.

For any positive integer p, let

$$L = D^p + \beta_1 D^{p-1} + \beta_2 D^{p-2} + \ldots + \beta_{p-1} D, \quad D \sim \frac{d}{dt}, \tag{13}$$

be a pth-order linear differential operator with constant coefficients $\beta_1, \beta_2, \ldots, \beta_{p-1}$. Let $\{A_n\}_{-\infty}^{\infty}$, $\{B_n\}_{-\infty}^{\infty}$, $\{C_n\}_{-\infty}^{\infty}$ be real sequences which satisfy (10). Define the sequence of linear operators $\{L_n\}_{-\infty}^{\infty}$ by

$$L_n[x(t)] = A_n L[x(t)] - B_n x(t), \quad -\infty<n<\infty. \tag{14}$$

Definition 6. A *symetrically coupled* infinite initial-value problem is a system of differential equations of the form

$$\begin{gathered} x_{-n-1} + L_{-n}[x_{-n}] + C_{-n} x_{-n+1} = 0 \ (n \geq 1) \\ x_{-1} + L_0[x_0] + x_1 = 0 \\ C_n x_{n-1} + L_n[x_n] + x_{n+1} = 0 \ (n \geq 1) \end{gathered} \tag{15}$$

with the initial conditions

$$\begin{gathered} x_k(0) = \alpha_0, \ \dot{x}_k(0) = \alpha_1, \ldots, x_k^{(p-1)}(0) = \alpha_{p-1} \ (k \text{ fixed}), \\ x_n(0) = 0, \ \dot{x}_n(0) = 0, \ldots, x_n^{(p-1)}(0) = 0, \ n \neq k, \ -\infty<n<\infty. \end{gathered} \tag{15.1}$$

It should be noted that the initial-value problems (11)-(11.1) and (12)-(12.1) are symmetrically coupled infinite initial-value problems in which the associated operators L are of order p=1 and p=2, respectively.

A solution of the symmetrically coupled infinite initial-value problem (15)-(15.1) may be obtained by decomposing the differential equations (15) and the initial conditions (15.1) into two half-infinite systems as follows.

Definition 7. Let $\{x_n\}$, $-\infty<n<\infty$, be any sequence of functions. The *symmetric part of the sequence* $\{x_n\}$ is the sequence $\{w_n\}$ given by

$$w_n(t) = \frac{x_n(t) + x_{-n}(t)}{2}, \quad n\geq 0, \tag{16}$$

and the *antisymmetric part of the sequence* $\{x_n\}$ is the sequence $\{z_n\}$ given by

$$z_n(t) = \frac{x_n(t) - x_{-n}(t)}{2}, \quad n\geq 0. \tag{17}$$

Addition of the equations in (15) which contain $L_n[x_n]$ and $L_{-n}[x_{-n}]$ ($n\geq 1$) and use of (16) yields the half-infinite system

$$\begin{gathered} \frac{1}{2} L_0[w_0] + w_1 = 0 \\ C_n w_{n-1} + L_n[w_n] + w_{n+1} = 0, \quad n\geq 1, \end{gathered} \tag{18}$$

with the initial conditions

$$w_k(0) = (1+\delta_{0k})\,\frac{\alpha_0}{2}\,,\ \dot{w}_k(0) = (1+\delta_{0k})\,\frac{\alpha_1}{2}\,,\ldots,\ w_k^{(p-1)}(0) = (1+\delta_{0k})\,\frac{\alpha_{p-1}}{2}\,,$$

$$w_n(0) = 0,\ \dot{w}_n(0) = 0,\ldots,\ w_n^{(p-1)}(0) = 0,\ n\neq k,\ n\geq 0. \tag{18.1}$$

Similarly, subtracting the equations in (15) which contain $L_n[x_n]$ and $L_{-n}[x_{-n}]$ $(n\geq 1)$ and using (17) yields the half-infinite system

$$z_0 = 0$$

$$L_1[z_1] + z_2 = 0 \tag{19}$$

$$C_n z_{n-1} + L_n[z_n] + z_{n+1} = 0,\ n\geq 2,$$

with the initial conditions

$$z_k(0) = (1-\delta_{0k})\,\frac{\alpha_0}{2}\,,\ \dot{z}_k(0) = (1-\delta_{0k})\,\frac{\alpha_1}{2}\,,\ldots,\ z_k^{(p-1)}(0) = (1-\delta_{0k})\,\frac{\alpha_{p-1}}{2}\,,$$

$$z_n(0) = 0,\ \dot{z}_n(0) = 0,\ldots,\ z_n^{(p-1)}(0) = 0,\ n\neq k,\ n\geq 0. \tag{19.1}$$

•

Lemma 4. A necessary and sufficient condition for a sequence of functions $\{x_n\}$, $-\infty<n<\infty$, to be a solution of the symmetrically coupled infinite initial-value problem (15)-(15.1) is that there exist sequences $\{w_n\}$ $(n\geq 0)$ and $\{z_n\}$ $(n\geq 0)$ which are solutions of the half-infinite initial-value problems (18)-(18.1) and (19)-(19.1), respectively.

Proof. This result follows from the linearity of L_n and the relations (16) and (17).

The coefficients A_n $(n\geq 0)$, B_n $(n\geq 0)$, C_n $(n\geq 1)$, from the differential equations (15) are used to define two sequences $\{R_n\}$ $(n\geq 0)$ and $\{Q_n\}$ $(n\geq 0)$ of polynomials generated by the recurrence relations

$$R_0 = 1$$

$$R_1(x) = \frac{A_0}{2} x + \frac{B_0}{2} \tag{20}$$

$$R_{n+1}(x) = (A_n x+B_n)R_n(x) - C_n R_{n-1}(x),\ n\geq 1,$$

and

$$Q_0 = 0,\ Q_1 = 1$$

$$Q_2(x) = A_1 x + B_1 \tag{21}$$

$$Q_{n+1}(x) = (A_n x+B_n)Q_n(x) - C_n Q_{n-1}(x),\ n\geq 2.$$

Note that $R_n(x)$ has degree exactly n, and $Q_n(x)$ $(n\geq 1)$ has degree exactly (n-1). The polynomials $\{R_n\}$ and $\{Q_n\}$ are now used in a separation technique for solving the differential equations of the half-infinite systems (18) and (19).

<u>Lemma 5</u>. The half-infinite systems (18) and (19) have solutions $\{w_n\}$ and $\{z_n\}$, respectively, of the form

$$w_n(t) = R_n(x)u(x,t), \quad n\geq 0, \tag{22}$$

and

$$z_n(t) = Q_n(x)u(x,t), \quad n\geq 0, \tag{23}$$

if and only if

$$L[u(x,t)] + xu(x,t) = 0 \tag{24}$$

for each x, where L is given by (13).

Proof. By substituting (22) into the equations (18) and using the recurrence (20), the (n+1)th equation of (18) becomes $A_n R_n(x)\{L[u(x,t)] + xu(x,t)\} = 0$, $n\geq 0$, for each x. Since $A_n \neq 0$, and since there exists no x so that $R_n(x) = 0$ for all $n\geq 1$, the assertion for $w_n(t)$ follows. The assertion for $z_n(t)$ can be verified similarly by using recurrence (21).

Conditions (2) are satisfied for each of the recurrences (20) and (21). Hence by Theorem 1 the polynomials $\{R_n\}$ are orthogonal on some interval [a,b] with respect to some normalized integrator α, and the polynomials $\{Q_n\}$ are orthogonal on some interval [c,d] with respect to some normalized integrator ω. These orthogonality properties are used to satisfy the initial conditions (18.1) and (19.1).

<u>Theorem 3</u>. Suppose that in (18) $\frac{C_n}{A_n A_{n-1}} > 0$ for $n=1,2,3,\ldots$. Let $\{R_n\}$ be the polynomials generated by (20), which are orthogonal on $[a,b]$ with respect to α, and let $\{Q_n\}$ be the polynomials generated by (21), which are orthogonal on $[c,d]$ with respect to ω. For L given by (13), let $u(x,t)$ be the solution of $L[u(x,t)] + xu(x,t) = 0$ which satisfies the initial conditions

$$u(x,0) = (1+\delta_{0k})\frac{\alpha_0}{2}\,,\ \frac{\partial u}{\partial t}(x,0) = (1+\delta_{0k})\frac{\alpha_1}{2}\,,\ldots,\ \frac{\partial^{p-1}u}{\partial t^{p-1}}(x,0)$$

$$= (1+\delta_{0k})\frac{\alpha_{p-1}}{2}\ ;$$

and let $v(x,t)$ be the solution of $L[v(x,t)] + xv(x,t) = 0$ which satisfies the initial conditions

$$v(x,0) = (1-\delta_{0k})\frac{\alpha_0}{2}\,,\ \frac{\partial v}{\partial t}(x,0) = (1-\delta_{0k})\frac{\alpha_1}{2}\,,\ldots,\ \frac{\partial^{p-1}v}{\partial t^{p-1}}(x,0)$$

$$= (1-\delta_{0k})\frac{\alpha_{p-1}}{2}\ .$$

Let

$$\gamma_0 = 1,\ \gamma_n = \int_a^b R_n^2(x)\,d\alpha(x) = \frac{A_0}{2A_n} C_1C_2\ldots C_n \quad (n\geq 1),$$

$$\xi_0 = \xi_1 = 1,\ \xi_n = \int_c^d Q_n^2(x)\,d\omega(x) = \frac{A_1}{A_n} C_2C_3\ldots C_n \ (n\geq 2).$$

For each non-negative integer n define,

$$w_n(t) = \int_a^b R_n(x)\,\frac{R_k(x)}{\gamma_k}\,u(x,t)d\alpha(x), \tag{25}$$

$$z_n(t) = \int_c^d Q_n(x)\,\frac{Q_k(x)}{\xi_k}\,v(x,t)d\omega(x). \tag{26}$$

Suppose that for each $t\geq 0$ and $j=1,2,\ldots,p$,

$$\frac{d^j w_n(t)}{dt^j} = \int_a^b R_n(x)\,\frac{R_k(x)}{\gamma_k}\,\frac{\partial^j u}{\partial t^j}(x,t)d\alpha(x) \tag{27.1}$$

and

$$\frac{d^j z_n(t)}{dt^j} = \int_c^d Q_n(x)\frac{Q_k(x)}{\xi_k}\,\frac{\partial^j v}{\partial t^j}(x,t)d\omega(x). \tag{27.2}$$

Then

(a) $\{w_n(t)\}$ is a solution of the initial-value problem (18)-(18.1);

(b) $\{z_n(t)\}$ is a solution of the initial-value problem (19)-(19.1).

Proof. From hypothesis (27.1) and the orthogonality of the polynomials $\{R_n\}$, a straightforward calculation shows that

$$\frac{d^j w_n(0)}{dt^j} = \int_a^b R_n(x)\,\frac{R_k(x)}{\gamma_k}\,\frac{\partial^j u}{\partial t^j}(x,0)d\alpha(x) = (1+\delta_{0k})\,\frac{\alpha_j}{2}\,\delta_{nk},$$

and the $\{w_n\}$ satisfy the initial conditions (18.1). That the $\{w_n\}$ satisfy the differential equations (18) follows from Lemma 5, the choice of $u(x,t)$ and the linearity of the integral. This proves assertion (a). For conclusion (b), note that $z_0(t) \equiv 0$, and for each $n \geq 1$ the hypothesis (27.2) and orthogonality of $\{Q_n\}$ yield the result

$$\frac{d^j z_n}{dt^j}(0) = \int_c^d Q_n(x) \frac{Q_k(x)}{\xi_k} \frac{\partial^j v}{\partial t^j}(x,0)d\omega(x) = (1-\delta_{0k}) \frac{\alpha_j}{2} \delta_{nk}.$$

Thus the $\{z_n\}$ satisfy the initial conditions (19.1). By Lemma 5 and the choice of $v(x,t)$, the $\{z_n\}$ clearly satisfy the differential equations (19). This completes the proof.

Theorem 3 may now be utilized to solve the first-order symmetrically coupled initial-value problem (11)-(11.1) and the second-order symetrically coupled initial-value problem (12)-(12.1). For the first-order system (11) the appropriate linear operator L is $L[y] = \frac{dy}{dt}$, and the required initial conditions are $x_k(0) = a_k$, $x_n(0) = 0$, $n \neq k$, $-\infty < n < \infty$. In accordance with Theorem 3, a routine calculation shows that if $k=0$, then $u(x,t) = a_0 e^{-xt}$ and $v(x,t) = 0$; and if $k \neq 0$, then $u(x,t) = v(x,t) = \frac{a_k}{2} e^{-xt}$. These remarks may be summarized to yield the following result.

<u>Theorem 4</u>. Suppose that in (11) $\frac{C_n}{A_n A_{n-1}} > 0$ for $n=1,2,3,\ldots$, and that the hypotheses (27.1) and (27.2) are satisfied. Then

(a) if $k=0$,

$$x_n(t) = x_{-n}(t) = a_0 \int_a^b R_n(x) e^{-xt} d\alpha(x), \; n\geq 0 \tag{28.1}$$

is a solution of the initial-value problem (11)-(11.1).

(b) If $k\neq 0$,

$$x_{-n}(t) = \frac{a_k}{2} \left[\int_a^b R_n(x) \frac{R_k(x)}{\gamma_k} e^{-xt} d\alpha(x) - \int_c^d Q_n(x) \frac{Q_k(x)}{\xi_k} e^{-xt} d\omega(x) \right], \; n\geq 1,$$

$$x_n(t) = \frac{a_k}{2} \left[\int_a^b R_n(x) \frac{R_k(x)}{\gamma_k} e^{-xt} d\alpha(x) + \int_c^d Q_n(x) \frac{Q_k(x)}{\xi_k} e^{-xt} d\omega(x) \right], \; n\geq 0, \tag{28.2}$$

is a solution of the initial-value problem (11)-(11.1).

Proof. For any non-negative integer k, Theorem 3 prescribes the form of $w_n(t)$ $(n\geq 0)$ and $z_n(t)$ $(n\geq 0)$. From relations (16) and (17), for each $n\geq 0$, $x_n(t) = w_n(t) + z_n(t)$ and $x_{-n}(t) = w_n(t) - z_n(t)$; and Lemma 4 ensures that $\{x_n\}$, $-\infty<n<\infty$, is a solution of (11)-(11.1).

For the second-order system (12) the appropriate linear operator L is

$$L[y] = \frac{d^2y}{dt^2} + \beta \frac{dy}{dt} ,$$

and the required initial conditions are $x_k(0) = a_k$, $\dot{x}_k(0) = b_k$, $x_n(0) = 0$, $\dot{x}_n(0) = 0$, $n \neq k$, $-\infty < n < \infty$. A straightforward calculation shows that if $k=0$ one should choose

$$u(x,t) = e^{-\frac{\beta t}{2}} \left[a_0 \cos \sqrt{x - \frac{\beta^2}{4}}\, t + \left(\frac{a_0 \beta}{2} + b_0 \right) \frac{\sin \sqrt{x - \frac{\beta^2}{4}}\, t}{\sqrt{x - \frac{\beta^2}{4}}} \right]$$

and $v(x,t) \equiv 0$, and if $k \neq 0$ one should choose

$$u(x,t) = v(x,t) = e^{-\frac{\beta t}{2}} \left[a_k \cos \sqrt{x - \frac{\beta^2}{4}}\, t + \left(\frac{a_k \beta}{2} + b_k \right) \frac{\sin \sqrt{x - \frac{\beta^2}{4}}\, t}{\sqrt{x - \frac{\beta^2}{4}}} \right].$$

From these remarks follows

<u>Theorem 5</u>. Suppose that in (12) $\frac{C_n}{A_n A_{n-1}} > 0$ for $n=1,2,3,\ldots$ and that the hypotheses (27.1) and (27.2) are satisfied. Then

(a) if $k=0$,

$$x_n(t) = x_{-n}(t) = \int_a^b R_n(x) F_0(x,t)\, d\alpha(x), \ n \geq 0, \tag{29.1}$$

is a solution of the initial-value problem (12)-(12.1).

(b) If $k \neq 0$,

$$x_{-n}(t) = \frac{1}{2}\left[\int_a^b R_n(x)\,\frac{R_k(x)}{\gamma_k}\,F_k(x,t)\,d\alpha(x)\right.$$

$$\left. - \int_c^d Q_n(x)\,\frac{Q_k(x)}{\xi_k}\,F_k(x,t)\,d\omega(x)\right], \quad n\geq 1,$$

(29.2)

$$x_n(t) = \frac{1}{2}\left[\int_a^b R_n(x)\,\frac{R_k(x)}{\gamma_k}\,F_k(x,t)\,d\alpha(x)\right.$$

$$\left. + \int_c^d Q_n(x)\,\frac{Q_k(x)}{\xi_k}\,F_k(x,t)\,d\omega(x)\right], \quad n\geq 0,$$

is a solution of the initial-value problem (12)-(12.1), where

$$F_k(x,t) = e^{-\frac{\beta t}{2}}\left[a_k \cos\sqrt{x - \frac{\beta^2}{4}}\;t + \left\{\frac{a_k\beta}{2} + b_k\right\}\frac{\sin\sqrt{x - \frac{\beta^2}{4}}\;t}{\sqrt{x - \frac{\beta^2}{4}}}\right], \quad k\geq 0.$$

CHAPTER III

TWO-DIMENSIONAL ARRAYS

The first section of this chapter contains a quadrature formula for numerical evaluation of double integrals. The development of the basic formula is contained in Appendix A, and Theorem 6 of the first section is a restatement of the result in terms of recurrence coefficients of two sets of orthogonal polynomials. The remainder of the chapter deals with the solution of initial-value problems associated with two-dimensional planar arrays. For each infinite system of differential equations considered, a brief description of an associated physical system is given. Solutions are given for both first- and second-order systems which correspond to arrays which cover a quarter-plane, a half-plane and the entire plane. Since large finite systems are also of some interest, it is shown that solutions of finite systems which are truncations of the infinite systems may be obtained by applying the quadrature formula of Theorem 6 to the components of the solutions of the infinite systems. Error estimates are given for the comparison of components of the corresponding infinite and finite systems.

The Quadrature Formula

Associated with an interval $[a,b]$ and a normalized integrator α on $[a,b]$, there exists a sequence of orthogonal polynomials $\{P_n\}$ which satisfy a three-term recurrence having the form (1). An easy induction proof shows that the polynomials $\phi_0 = P_0$,

$$\phi_n = \frac{P_n}{A_0 A_1 \ldots A_{n-1}}, \quad n \geq 1,$$

are monic polynomials which satisfy the recurrence

$$\phi_0 = 1$$

$$\phi_1(x) = x + b_0$$

$$\phi_{n+1}(x) = (x+b_n)\phi_n(x) - c_n \phi_{n-1}(x), \quad n \geq 1,$$

where

$$b_n = \frac{B_n}{A_n}, \quad n \geq 0, \qquad c_n = \frac{C_n}{A_n A_{n-1}}, \quad n \geq 1.$$

Clearly the polynomials $\{\phi_n\}$ are also orthogonal on $[a,b]$ with respect to α and the zeros of ϕ_n and P_n agree.

Let $\alpha(x)$ and $\omega(y)$ be normalized integrators defined on the intervals $a \leq x \leq b$ and $c \leq y \leq d$, respectively, and let $\{P_n(x)\}$ and $\{Q_n(y)\}$ be, respectively, the sequences of polynomials orthogonal on $[a,b]$ with respect to α and on $[c,d]$ with respect to ω. Then $\{P_n\}$ satisfy a recurrence

$$P_0 = 1$$

$$P_1(x) = A_0x + B_0 \tag{30}$$

$$P_{n+1}(x) = (A_nx+B_n)P_n(x) - C_nP_{n-1}(x),\ n\geq 1,$$

and $\{Q_n\}$ satisfy a recurrence

$$Q_0 = 1$$

$$Q_1(y) = D_0y + E_0 \tag{31}$$

$$Q_{n+1}(y) = (D_ny+E_n)Q_n(y) - F_nQ_{n-1}(y),\ n\geq 1.$$

The polynomials $\phi_0 = 1$, $\phi_n = \dfrac{P_n}{A_0A_1\ldots A_{n-1}}$, $n\geq 1$, and $\psi_0 = 1$, $\psi_n = \dfrac{Q_n}{D_0D_1\ldots D_{n-1}}$, $n\geq 1$, are, respectively, the monic polynomials associated with α on $[a,b]$ and ω on $[c,d]$.

Let N and M be any two positive integers; $x_1,x_2,\ldots,x_N$ the N zeros of $P_N(x)$; and $y_1,y_2,\ldots,y_M$ the M zeros of $Q_M(y)$. For $i=1,2,\ldots,N$ let $\{\lambda_i\}$ denote the Christoffel numbers associated with the zeros of $P_N(x)$; i.e., in terms of the monic polynomials ϕ_n,

$$\lambda_i = \frac{c_0c_1\cdots c_{N-1}}{\phi_{N-1}(x_i)\phi_N'(x_i)},\quad 1\leq i\leq N,$$

where $c_0=1$ by definition.

For $j=1,2,\ldots,M$ let $\{\kappa_j\}$ denote the Christoffel numbers associated with the zeros of $Q_M(y)$; i.e.,

$$\kappa_j = \frac{f_0 f_1 \ldots f_{M-1}}{\psi_{M-1}(y_j)\psi_M'(y_j)}\ , \quad 1 \le j \le M,$$

where $f_n = \dfrac{F_n}{D_n D_{n-1}}$, $n \ge 1$, and $f_0 = 1$ by definition.

Let

$$\tau_N = \int_a^b \phi_N^2(x)\,d\alpha(x)$$

and

$$\sigma_M = \int_c^d \psi_M^2(y)\,d\omega(y).$$

Note that in terms of the coefficients in the recurrence relation (30)

$$\tau_N = \int_a^b \phi_N^2(x)\,d\alpha(x) = \frac{1}{[A_0 A_1 \ldots A_{N-1}]^2} \int_a^b P_N^2(x)\,d\alpha(x)$$

$$= \frac{1}{[A_0 A_1 \ldots A_{N-1}]^2}\ \frac{A_0}{A_N}\ C_0 C_1 \ldots C_N$$

$$= \frac{C_0 C_1 \ldots C_N}{A_0 A_1^2 \ldots A_{N-1}^2 A_N}\ ;$$

similarly from recurrence (31)

$$\sigma_M = \frac{F_0 F_1 \ldots F_M}{D_0 D_1^2 \ldots D_{M-1}^2 D_M}\ .$$

Suppose that $f(x,y)$, $\dfrac{\partial^{2N} f}{\partial x^{2N}}(x,y)$, and $\dfrac{\partial^{2M} f}{\partial y^{2M}}(x,y)$ are continuous on $[a,b] \times [c,d]$; then by Theorem A.1, Appendix A,

$$\int_a^b \int_c^d f(x,y)d\omega(y)d\alpha(x) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_i \kappa_j f(x_i,y_j) \tag{32}$$

$$+ \frac{1}{(2N)!} \frac{\partial^{2N} f}{\partial x^{2N}} (\hat{x}_1,\hat{y}_1)\tau_N + \frac{1}{(2M)!} \frac{\partial^{2M} f}{\partial y^{2M}} (\hat{x}_2,\hat{y}_2)\sigma_M$$

for some $\hat{x}_1,\hat{x}_2 \epsilon [a,b]$ and $\hat{y}_1,\hat{y}_2 \epsilon [c,d]$. The reformulation of (32) in terms of the recurrence coefficients for the polynomials leads to

Theorem 6 (The Quadrature Formula). Suppose that condition (2) holds for each of the recurrences (30) and (31), so that $\{P_n\}$ are orthogonal on $[a,b]$ with respect to a normalized integrator $\alpha(x)$ and $\{Q_n\}$ are orthogonal on $[c,d]$ with respect to a normalized integrator $\omega(y)$. Let N and M be any two positive integers, and let $x_1,x_2,\ldots,x_N$ and $y_1,y_2,\ldots,y_M$ be, respectively, the zeros of $P_N(x)$ and $Q_M(y)$. For $i=1,2,\ldots,N$ let

$$\lambda_i = \frac{A_0 C_0 C_1 \ldots C_{N-1}}{P_{N-1}(x_i)P_N'(x_i)}, \quad C_0 \stackrel{\Delta}{=} 1,$$

and for $j=1,2,\ldots,M$ let

$$\kappa_j = \frac{D_0 F_0 F_1 \ldots F_{M-1}}{Q_{M-1}(y_j)Q_M'(y_j)}, \quad F_0 \stackrel{\Delta}{=} 1.$$

Let $f(x,y)$, $\frac{\partial^{2N} f}{\partial x^{2N}}(x,y)$, and $\frac{\partial^{2M} f}{\partial y^{2M}}(x,y)$ be continuous on $[a,b] \times [c,d]$. Then there exist $\hat{x}_1,\hat{x}_2 \epsilon [a,b]$ and $\hat{y}_1,\hat{y}_2 \epsilon [c,d]$ such that

$$\int_a^b \int_c^d f(x,y)d\omega(y)d\alpha(x) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_i \kappa_j f(x_i,y_j) + E, \tag{33}$$

where

$$E = \frac{1}{(2N)!} \frac{C_1 C_2 \cdots C_N}{A_0 A_1^2 \cdots A_{N-1}^2 A_N} \frac{\partial^{2N} f}{\partial x^{2N}} (\hat{x}_1,\hat{y}_1)$$

$$+ \frac{1}{(2M)!} \frac{F_1 F_2 \cdots F_M}{D_0 D_1^2 \cdots D_{M-1}^2 D_M} \frac{\partial^{2M} f}{\partial y^{2M}} (\hat{x}_2,\hat{y}_2).$$

Quarter-Planar Arrays

This section deals with the formulation and solution of two pairs of countable systems of differential equations associated with quarter-planar arrays. The first pair is an infinite first-order system and its finite truncation; the second is an infinite second-order system and its finite truncation.

Let $\{\rho_i\}$ $(i\geq 0)$ and $\{\mu_j\}$ $(j\geq 0)$ be sequences of real constants such that $\rho_0\geq 0$, $\rho_i>0$, $i\geq 1$, and $\mu_0\geq 0$, $\mu_j>0$, $j>1$. For any two positive sequences $\{a_i\}$ $(i\geq 0)$ and $\{b_j\}$ $(j\geq 0)$ and any positive constant m define

$$m_{ij} = m a_i b_j, \quad i\geq 0,\ j\geq 0,$$

$$\rho_{ij} = \rho_i b_j, \quad i\geq 0,\ j\geq 0, \tag{34}$$

$$\mu_{ij} = a_i \mu_j, \quad i\geq 0,\ j\geq 0.$$

For any two integers $N \geq 2$ and $M \geq 2$, let p and q be non-negative integers which are, respectively, less than N and M, and let a_{pq} be a specified constant. Using the coefficients (34), construct the infinite first-order system

$$m_{ij}\dot{y}_{ij} = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij}) \tag{35.1}$$

$$+ \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij}) \quad (i \geq 0,\ j \geq 0)$$

and its finite N-by-M truncation

$$m_{ij}\dot{y}_{ij} = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij}) \tag{35.2}$$

$$+ \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij}),$$

$$0 \leq i \leq N-1,\ 0 \leq j \leq M-1,$$

both subject to the initial conditions

$$y_{pq}(0) = a_{pq}, \tag{36}$$

$$y_{ij}(0) = 0, \quad i \neq p,\ j \neq q.$$

For the sake of notational brevity, the conventions $y_{-1,j}(t) \equiv 0$ $(j \geq 0)$ and $y_{i,-1}(t) \equiv 0$ $(i \geq 0)$ are to be used in (35.1) and (35.2), and the conventions $y_{i,M}(t) \equiv 0$ $(0 \le i \le N-1)$ and $y_{N,j}(t) \equiv 0$ $(0 \le j \le M-1)$ are to be used in (35.2). The initial-value problem (35.1), (36) may be interpreted as the equations of motion, in terms of velocities, of a quarter-infinite array of sliding plates arranged in rows and columns. Each inertial element of the array is coupled by viscous friction to each of its nearest neighbors, and velocities are normal to the plane of the array (see Figure 4).

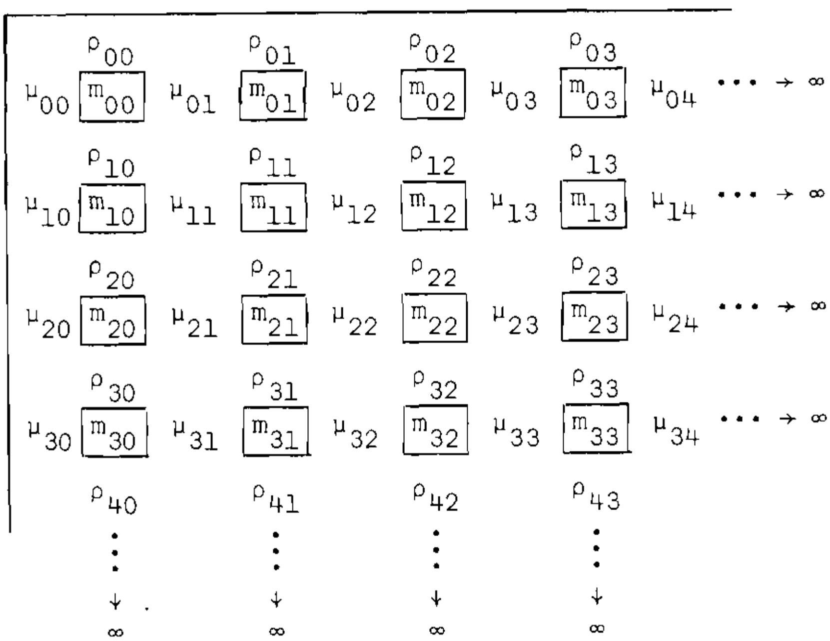

Figure 4. An Infinite Quarter-Planar Array of Sliding Plates

The initial-value problem (35.2), (36) may be interpreted as the equations of motion of a finite system obtained by deleting all but the first N rows and first M columns of inertial elements of the above infinite system. Note that in the finite truncation, the Nth row and

Mth column of inertial elements are to be thought of as coupled to a fixed wall. Also in both the infinite system and its finite truncation, a free edge condition may be accommodated at the top edge by choosing $\rho_0 = 0$ and at the left edge by choosing $\mu_0 = 0$.

For any two integers $N \geq 2$ and $M \geq 2$, let p and q be non-negative integers which are, respectively, less than N and M; let a_{pq} and b_{pq} be two constants; and let $\beta \geq 0$ be a constant. Using the coefficients (34), consider the infinite second-order system

$$m_{ij}(\ddot{y}_{ij}+\beta\dot{y}_{ij}) = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij}) + \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij}) \quad (i \geq 0,\ j \geq 0) \tag{37.1}$$

and its finite N-by-M truncation

$$m_{ij}(\ddot{y}_{ij}+\beta\dot{y}_{ij}) = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij}) + \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij}), \quad 0 \leq i \leq N-1,\ 0 \leq j \leq M-1, \tag{37.2}$$

both subject to the initial conditions

$$y_{pq}(0) = a_{pq}, \quad \dot{y}_{pq}(0) = b_{pq}, \tag{38}$$

$$y_{ij}(0) = 0, \quad \dot{y}_{ij}(0) = 0, \quad i \neq p, \ j \neq q.$$

The notational conventions $y_{-1,j}(t) \equiv 0$ $(j \geq 0)$ and $y_{i,-1}(t) \equiv 0$ $(i \geq 0)$ are to be used in (37.1) and (37.2), and the conventions $y_{N,j}(t)$ $(0 \leq j \leq M-1)$, $y_{i,M}(t) \equiv 0$ $(0 \leq i \leq N-1)$ are to be used in (37.2).

The differential equations (37.1) may be viewed as the linearized equations of motion, in terms of displacements, of an infinite array of damped oscillators. Each inertial element is coupled by a linear restoring force to each of its nearest neighbors and is subjected to a damping force βm_{ij} proportional to its mass (see Figure 5).

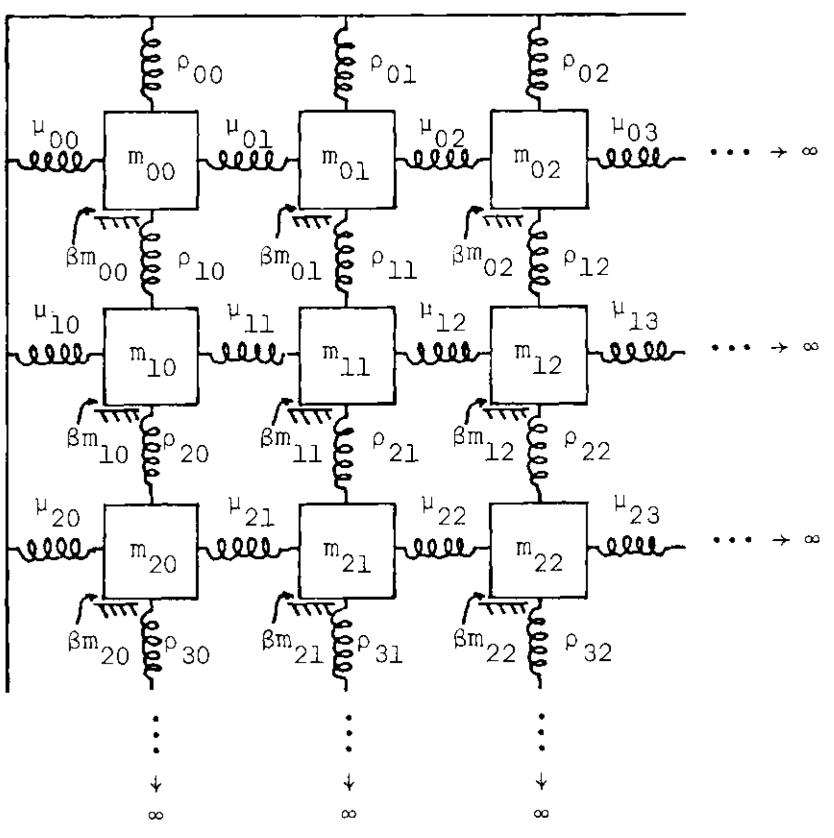

Figure 5. An Infinite Quarter-Planar Array of Damped Oscillators

Motion is normal to the plane of the array. The differential equations (37.2) may be interpreted as the equations of motion of the finite system obtained by deleting all but N rows and M columns from the above infinite system.

The coefficients (34) are used to generate two sequences of orthogonal polynomials which in turn are used to construct a solution of the infinite initial-value problems (35.1), (36) and (37.1), (38). By use of the quadrature formula (33), the corresponding solutions of the finite initial-value problems (35.2), (36) and (37.2), (38) are obtained.

From the coefficients (34), define

$$\begin{aligned}
A_n &= -\frac{m_{nj}}{\rho_{n+1,j}} = -\frac{ma_n}{\rho_{n+1}} \quad (n\geq 0), \\
B_n &= \frac{\rho_{nj} + \rho_{n+1,j}}{\rho_{n+1,j}} = 1 + \frac{\rho_n}{\rho_{n+1}} \quad (n\geq 0), \\
C_n &= \frac{\rho_{nj}}{\rho_{n+1,j}} = \frac{\rho_n}{\rho_{n+1}} \quad (n\geq 1),
\end{aligned} \tag{39.1}$$

and

$$\begin{aligned}
D_n &= -\frac{m_{in}}{\mu_{i,n+1}} = -\frac{mb_n}{\mu_{n+1}} \quad (n\geq 0), \\
E_n &= \frac{\mu_{in} + \mu_{i,n+1}}{\mu_{i,n+1}} = 1 + \frac{\mu_n}{\mu_{n+1}} \quad (n\geq 0), \\
F_n &= \frac{\mu_{in}}{\mu_{i,n+1}} = \frac{\mu_n}{\mu_{n+1}} \quad (n\geq 1).
\end{aligned} \tag{39.2}$$

Let $\{P_n\}$ be the polynomials generated by (30) with the coefficients (39.1) and $\{Q_n\}$ the polynomials generated by (31) with the coefficients (39.2).

<u>Lemma 6</u>. The differential equations (35.1) have a solution of the form

$$y_{ij}(t) = P_i(x)Q_j(y)u(x,y,t), \quad i\geq 0,\ j\geq 0,$$

if and only if

$$\frac{\partial u}{\partial t} + (x+y)u = 0.$$

Proof. By substituting the assumed form for y_{ij} into (35.1) and using recurrences (30) and (31) and the expressions (39.1) and (39.2), one finds that the differential equations (35.1) reduce to

$$ma_ib_jP_i(x)Q_j(y)\left\{\frac{\partial u}{\partial t} + (x+y)u\right\} = 0, \quad i\geq 0,\ j\geq 0.$$

Since this equality must hold for all $i\geq 0$, $j\geq 0$ and all x, y, the conclusion follows at once.

Lemma 6 shows that a solution of the differential equations (35.1) may be found by the separation technique indicated. It remains to satisfy the initial conditions (36).

For each integer $n\geq 0$, define

$$\gamma_n = \int_a^b P_n^2(x)d\alpha(x), \quad \xi_n = \int_c^d Q_n^2(y)d\omega(y),$$

and for each pair of integers $i\geq 0$, $j\geq 0$ define

$$f_{ij}(x,y,t) = P_i(x)Q_j(y)F(x,y,t), \tag{40}$$

where

$$F(x,y,t) = a_{pq} \frac{P_p(x)}{\gamma_p} \frac{Q_q(y)}{\xi_q} e^{-(x+y)t}.$$

Theorem 7. Suppose that the coefficients in (39.1) and (39.2) satisfy condition (2)--that is, the P_n ($n\geq 0$) given by (30) are orthogonal on $[a,b]$ with respect to $\alpha(x)$, and the Q_n ($n\geq 0$) given by (31) are orthogonal on $[c,d]$ with respect to $\omega(y)$. For each $i\geq 0$, $j\geq 0$ let

$$V_{ij}(t) = \int_a^b \int_c^d f_{ij}(x,y,t)d\omega(y)d\alpha(x), \tag{41}$$

and for $0\leq i\leq N$, $0\leq j\leq M$ let

$$v_{ij}(t) = \sum_{r=1}^{N} \sum_{s=1}^{M} \lambda_r \kappa_s f_{ij}(x_r,y_s,t). \tag{42}$$

Suppose that for $i\geq 0$, $j\geq 0$ and each $t\geq 0$

$$\dot{V}_{ij}(t) = \int_a^b \int_c^d \frac{\partial f_{ij}}{\partial t}(x,y,t)d\omega(y)d\alpha(x). \tag{43}$$

Then

(a) for $i\geq 0$, $j\geq 0$ the sequence $\{V_{ij}\}$ defined by (41) is a solution of the infinite initial-value problem (35.1), (36);

(b) for $i=0,1,2,\ldots,N-1$, $j=0,1,\ldots,M-1$ the sequence $\{v_{ij}\}$ defined by (42) is a solution of the finite initial-value problem (35.2), (36);

(c) for $i=0,1,\ldots,N-1$, $j=0,1,\ldots,M-1$ and for each $t\geq 0$ there exist $\hat{x}_1,\hat{x}_2\epsilon[a,b]$ and $\hat{y}_1,\hat{y}_2\epsilon[c,d]$ such that

$$V_{ij}(t) = v_{ij}(t) + E_v,$$

where

$$E_v = \frac{1}{[(2N)!]}\,\frac{C_1C_2\ldots C_N}{A_0A_1^2\ldots A_{N-1}^2A_N}\,\frac{\partial^{2N}f_{ij}}{\partial x^{2N}}(\hat{x}_1,\hat{y}_1,t)$$

$$+\frac{1}{[(2M)!]}\,\frac{F_1F_2\ldots F_M}{D_0D_1^2\ldots D_{M-1}^2D_M}\,\frac{\partial^{2M}f_{ij}}{\partial y^{2M}}(\hat{x}_2,\hat{y}_2,t).$$

Proof. It is evident from the orthogonality of the polynomials $\{P_n\}$ and $\{Q_n\}$, that $\{V_{ij}(0)\}$ satisfies the initial conditions (36). Since $\frac{\partial F}{\partial t}(x,y,t) = -(x+y)F(x,y,t)$, an application of Lemma 6 shows that $\{V_{ij}(t)\}$ satisfies the differential equations (35.1), which proves assertion (a). To prove assertion (b), note that $f_{ij}(x,y,0) = a_{pq}P_i(x)\,\frac{P_p(x)}{\gamma_p}\,Q_j(y)\,\frac{Q_q(y)}{\xi_q}$ is a polynomial of degree $\leq(2N-2)$ in x and of degree $\leq(2M-2)$ in y. By the quadrature formula it follows that

$$V_{ij}(0) = \int_a^b \int_c^d f_{ij}(x,y,0)d\omega(y)d\alpha(x)$$

$$= \sum_{r=1}^{N} \sum_{s=1}^{M} \lambda_r \kappa_s f_{ij}(x_r,y_s,0) = v_{ij}(0).$$

Thus $\{v_{ij}\}$ satisfy the same initial conditions (36) as do $\{V_{ij}\}$. Note that $v_{iM}(t) \equiv 0$, $i=0,1,\ldots,N-1$, and $v_{Nj}(t) \equiv 0$, $j=0,1,\ldots,M-1$; then another application of Lemma 6 shows that $\{v_{ij}\}$ satisfy the system of differential equations (35.2). This proves assertion (b). Clearly, assertion (c) is simply a restatement of the quadrature formula (33) applied to $f_{ij}(x,y,t)$. This completes the proof.

The next theorem is the analog of Theorem 7 for the second-order initial-value problems (37.1), (38) and (37.2), (38). The proof of this theorem parallels the proof of Theorem 7 and the details are omitted.

Lemma 7. The differential equations (37.1) have a solution of the form

$$x_{ij}(t) = P_i(x)Q_j(y)u(x,y,t), \quad i\geq 0,\ j\geq 0,$$

if and only if

$$\frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} + (x+y)u = 0.$$

Proof. If the assumed form for x_{ij} is substituted into (37.1) and the recurrences (30) and (31) are used, the differential equations reduce to

$$ma_i b_j P_i(x)Q_j(y)\left\{\frac{\partial^2 u}{\partial t^2} + \beta\,\frac{\partial u}{\partial t} + (x+y)u\right\} = 0,$$

which must hold for i≥0, j≥0 and all x,y. The result follows from this equality.

For each pair of integers i≥0, j≥0 define

$$g_{ij}(x,y,t) = P_i(x)Q_j(y)G(x,y,t), \tag{44}$$

where

$$G(x,y,t) = e^{-\frac{\beta t}{2}}\,\frac{P_p(x)}{\gamma_p}\,\frac{Q_q(y)}{\xi_q}\left[a_{pq}\cos\sqrt{x+y-\frac{\beta^2}{4}}\,t\right.$$

$$\left.+\left(\frac{\beta a_{pq}}{2}+b_{pq}\right)\frac{\sin\sqrt{x+y-\frac{\beta^2}{4}}\,t}{\sqrt{x+y-\frac{\beta^2}{4}}}\right].$$

<u>Theorem 8</u>. Suppose that the coefficients in (39.1) and (39.2) satisfy condition (2)--that is, the P_n (n≥0) are orthogonal on [a,b] with respect to $\alpha(x)$, and the Q_n (n≥0) are orthogonal on [c,d] with respect to $\omega(y)$.

For each i≥0, j≥0 let

$$X_{ij}(t) = \int_a^b\int_c^d g_{ij}(x,y,t)\,d\omega(y)\,d\alpha(x), \tag{45}$$

and for $0 \le i \le N-1$, $0 \le j \le M-1$ let

$$x_{ij}(t) = \sum_{r=1}^{N} \sum_{s=1}^{M} \lambda_r \kappa_s g_{ij}(x_r, y_s, t). \tag{46}$$

Suppose that for $i \ge 0$, $j \ge 0$ and each $t \ge 0$

$$\begin{aligned} \dot{X}_{ij}(t) &= \int_a^b \int_c^d \frac{\partial g_{ij}}{\partial t}(x,y,t)\,d\omega(y)\,d\alpha(x), \\ \ddot{X}_{ij}(t) &= \int_a^b \int_c^d \frac{\partial^2 g_{ij}}{\partial t^2}(x,y,t)\,d\omega(y)\,d\alpha(x). \end{aligned} \tag{47}$$

Then:

(a) for $i \ge 0$, $j \ge 0$ the sequence $\{X_{ij}\}$ defined by (45) is a solution of the infinite initial-value problem (37.1), (38);

(b) for $i=0,1,\ldots,N-1$, $j=0,1,\ldots,M-1$ the sequence $\{x_{ij}\}$ defined by (46) is a solution of the finite initial-value problem (37.2), (38);

(c) for $i=0,1,\ldots,N-1$, $j=0,1,\ldots,M-1$ and for each $t \ge 0$ there exist $\hat{x}_1, \hat{x}_2 \in [a,b]$ and $\hat{y}_1, \hat{y}_2 \in [c,d]$ such that $X_{ij}(t) = x_{ij}(t) + E_x$,

where

$$E_x = \frac{1}{[(2N)!]} \frac{C_1 C_2 \cdots C_N}{A_0 A_1^2 \cdots A_{N-1}^2 A_N} \frac{\partial^{2N} g_{ij}}{\partial x^{2N}}(\hat{x}_1, \hat{y}_1, t) + \frac{1}{[(2M)!]} \frac{F_1 F_2 \cdots F_M}{D_0 D_1^2 \cdots D_{M-1}^2 D_M} \frac{\partial^{2M} g_{ij}}{\partial y^{2M}}(\hat{x}_2, \hat{y}_2, t).$$

Half-Planar Arrays

The separation technique* of the preceding section may be combined with the decomposition procedure** used in Chapter II for the one-dimensional infinite initial-value problem to obtain solutions for initial-value problems associated with two-dimensional arrays which cover a half-plane and have physical symmetry about a selected column of inertial elements.

Let $\{\rho_i\}$ $(i\geq 0)$ and $\{\mu_j\}$ $(j\geq 1)$ be sequences of real constants such that $\rho_o\geq 0$, $\rho_i>0$, $i\geq 1$, and $\mu_j>0$, $j\geq 1$. For any two positive sequences $\{a_i\}$ $(i\geq 0)$ and $\{b_j\}$ $(j\geq 0)$ and any positive constant m define

$$
\begin{aligned}
m_{ij} &= ma_i b_{|j|}, && i\geq 0,\ -\infty<j<\infty,\\
\rho_{ij} &= \rho_i b_j, && i\geq 1,\ j\geq 0, \qquad\qquad (48)\\
\mu_{ij} &= a_i \mu_{|j|}, && i\geq 0,\ -\infty<j<\infty,\ j\neq 0.
\end{aligned}
$$

For any two integers $N\geq 2$ and $M\geq 2$, let p and q be non-negative integers which are, respectively, less than N and M, and let a_{pq} be a given constant. Using the coefficients (48), construct the infinite initial-value problem

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*The separation method refers to the uncoupling of the differential equations by assuming a solution of the form $y_{ij}(t) = P_i(x)Q_j(y)u(x,y,t)$.

**The decomposition procedure refers to the resolution of the sequence $\{y_n\}$, $-\infty<n<\infty$, into its symmetric part $w_n = (y_n+y_{-n})/2$, $n\geq 0$, and antisymmetric part $z_n = (y_n-y_{-n})/2$, $n\geq 0$.

$$m_{i,-j}\dot{y}_{i,-j} = \mu_{i,-j}(y_{i,-j+1}-y_{i,-j}) + \mu_{i,-j-1}(y_{i,-j-1}-y_{i,-j})$$

$$+ \rho_{i,-j}(y_{i-1,-j}-y_{i,-j}) + \rho_{i+1,-j}(y_{i+1,-j}-y_{i,-j})$$

$$(i\geq 0,\ j\geq 1),$$

$$m_{i0}\dot{y}_{i0} = \mu_{i,-1}(y_{i,-1}-y_{i0}) + \mu_{i1}(y_{i1}-y_{i0}) \tag{49.1}$$

$$+ \rho_{i0}(y_{i-1,0}-y_{i0}) + \rho_{i+1,0}(y_{i+1,0}-y_{i0})\ (i\geq 0),$$

$$m_{ij}\dot{y}_{ij} = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij})$$

$$+ \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij})$$

$$(i\geq 0,\ j\geq 1)$$

and its finite truncation

$$m_{i,-j}\dot{y}_{i,-j} = \mu_{i,-j}(y_{i,-j+1}-y_{i,-j}) + \mu_{i,-j-1}(y_{i,-j-1}-y_{i,-j})$$

$$+ \rho_{i,-j}(y_{i-1,-j}-y_{i,-j}) + \rho_{i+1,-j}(y_{i+1,-j}-y_{i,-j})$$

$$(0\leq i\leq N-1,\ 1\leq j\leq M-1),$$

$$m_{i0}\dot{y}_{i0} = \mu_{i1}(y_{i1}-y_{i0}) + \mu_{i,-1}(y_{i,-1}-y_{i0}) \tag{49.2}$$

$$+ \rho_{i0}(y_{i-1,0}-y_{i0}) + \rho_{i+1,0}(y_{i+1,0}-y_{i0})$$

$$(0 \le i \le N-1),$$

$$m_{ij}\dot{y}_{ij} = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij})$$

$$+ \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij})$$

$$(0 \le i \le N-1,\ 1 \le j \le M-1),$$

both subject to the initial conditions

$$y_{pq}(0) = a_{pq},$$

$$y_{ij}(0) = 0,\ i \ne p,\ j \ne q. \tag{50}$$

In the systems (49.1) and (49.2) the convention $y_{-1,j}(t) \equiv 0$ $(-\infty<j<\infty)$ is to be used, and in the system (49.2) the conventions $y_{i,-M}(t) \equiv 0$ $(0 \le i \le N-1)$, $y_{i,M}(t) \equiv 0$ $(0 \le i \le N-1)$, and $y_{N,j}(t) \equiv 0$ $(-M+1 \le j \le M-1)$ are to be used. A physical system which may be associated with the initial-value problem (49.1), (50) is the half-planar array of sliding plates shown in Figure 6.

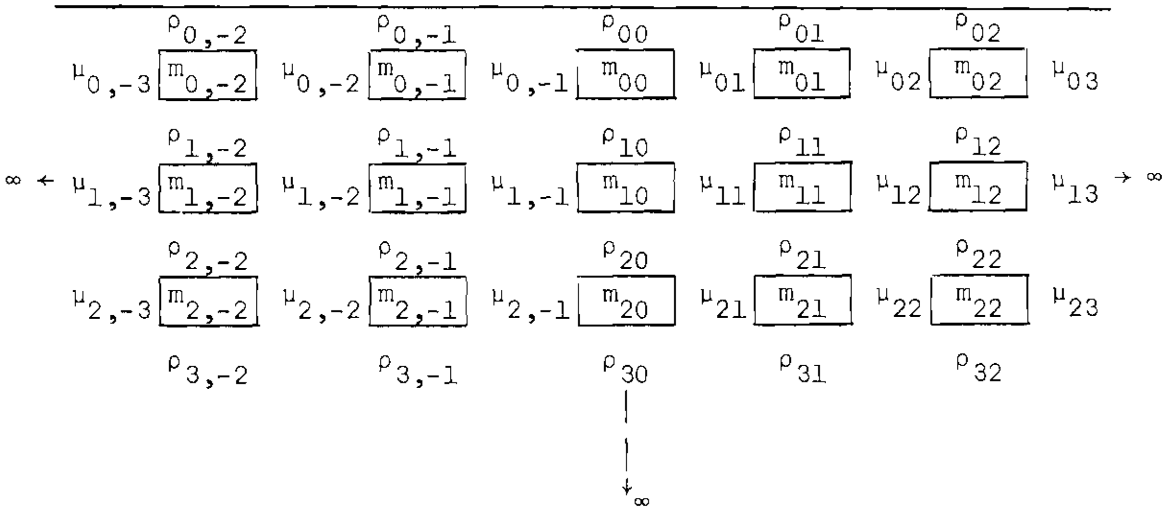


Figure 6. An Infinite Half-Planar Array of Sliding Plates

Each mass is coupled by viscous friction to each of its nearest neighbors, and $y_{ij}(t)$ ($i\geq 0$, $-\infty<j<\infty$) is to be interpreted as the velocity of mass m_{ij} normal to the plane of the array. A physical system associated with the finite initial-value problem (49.2), (50) is the array obtained from the above system by deleting all but the first N rows and all columns except the (2M-1) columns centered on column zero.

The two systems (49.1) and (49.2) may be decomposed into two equivalent systems of the type considered in the preceding section. Recall that a sequence of functions $\{y_{ij}\}$ ($i\geq 0$, $-\infty<j<\infty$) may be decomposed into a symmetric part $\{w_{ij}\}$ and an antisymmetric part $\{z_{ij}\}$, where

$$w_{ij}(t) = \frac{y_{ij}(t) + y_{i,-j}(t)}{2}, \quad i\geq 0,\ j\geq 0, \tag{51}$$

and

$$z_{ij}(t) = \frac{y_{ij}(t) - y_{i,-j}(t)}{2}, \quad i\geq 0,\ j\geq 0. \tag{52}$$

Addition of the equations which contain $m_{i,-j}\dot{y}_{i,-j}$ and $m_{ij}\dot{y}_{ij}$ ($i\geq 0$, $j\geq 1$) in (49.1) and (49.2) and use of (51) yields the infinite system

$$m_{i0}\dot{w}_{i0} = 2\mu_{i1}(w_{i1}-w_{i0}) + \rho_{i0}(w_{i-1,0}-w_{i0}) \tag{53.1}$$

$$+ \rho_{i+1,0}(w_{i+1,0}-w_{i0}), \quad i\geq 0,$$

$$m_{ij}\dot{w}_{ij} = \mu_{ij}(w_{i,j-1}-w_{ij}) + \mu_{i,j+1}(w_{i,j+1}-w_{ij}) + \rho_{ij}(w_{i-1,j}-w_{ij})$$

$$+ \rho_{i+1,j}(w_{i+1,j}-w_{ij}), \quad i\geq 0,\ j\geq 1,$$

with its finite truncation

$$m_{i0}\dot{w}_{i0} = 2\mu_{i1}(w_{i1}-w_{i0}) + \rho_{i0}(w_{i-1,0}-w_{i0}) + \rho_{i+1,0}(w_{i+1,0}-w_{i0}),$$

$$0\leq i\leq N-1,$$

$$m_{ij}\dot{w}_{ij} = \mu_{ij}(w_{i,j-1}-w_{ij}) + \mu_{i,j+1}(w_{i,j+1}-w_{ij}) \tag{53.2}$$

$$+ \rho_{ij}(w_{i-1,j}-w_{ij}) + \rho_{i+1,j}(w_{i+1,j}-w_{ij}),$$

$$0\leq i\leq N-1,\ 1\leq j\leq M-1,$$

both subject to the initial conditions

$$w_{pq}(0) = (1+\delta_{0q})\,\frac{a_{pq}}{2}$$

$$w_{ij}(0) = 0,\ i\neq p,\ j\neq q. \tag{54}$$

Similarly, subtracting the equations which contain $m_{i,-j}\dot{y}_{i,-j}$ and $m_{ij}\dot{y}_{ij}$ in (49.1) and (49.2) and using (52) yields the infinite system

$$z_{i0} = 0,$$

$$m_{ij}\dot{z}_{ij} = \mu_{ij}(z_{i,j-1}-z_{ij}) + \mu_{i,j+1}(z_{i,j+1}-z_{ij}) \tag{55.1}$$

$$+ \rho_{ij}(z_{i-1,j}-z_{ij}) + \rho_{i+1,j}(z_{i+1,j}-z_{ij}),\ i\geq 0,\ j\geq 1,$$

with its finite truncation

$$z_{i0} = 0,$$

$$m_{ij}\dot{z}_{ij} = \mu_{ij}(z_{i,j-1}-z_{ij}) + \mu_{i,j+1}(z_{i,j+1}-z_{ij}) \tag{55.2}$$

$$+ \rho_{ij}(z_{i-1,j}-z_{ij}) + \rho_{i+1,j}(z_{i+1,j}-z_{ij}),$$

$$0\leq i\leq N-1,\ 1\leq j\leq M-1,$$

both subject to the initial conditions

$$
\begin{aligned}
z_{pq}(0) &= (1-\delta_{0q})\frac{a_{pq}}{2}, \\
z_{ij}(0) &= 0, \quad i\neq p, \ j\neq q.
\end{aligned}
\tag{56}
$$

Lemma 8. A necessary and sufficient condition for a sequence of functions $\{y_{ij}\}$ to be a solution of the initial-value problem (49.1), (50) [or (49.2), (50)] is that there exist sequences $\{w_{ij}\}$ and $\{z_{ij}\}$ which are solutions of the initial-value problems (53.1), (54) [or (53.2), (54)] and (55.1), (56) [or (55.2), (56)], respectively.

Proof. This conclusion follows from the linearity of the differential equations and the definitions (51) and (52).

The coefficients in the differential equations (53.1) and (55.1) are used to generate three sequences of orthogonal polynomials which are subsequently used to construct solutions of the initial-value problems (53.1), (54) and (55.1), (56). These solutions are utilized to produce a solution of the infinite initial-value problem (49.1), (50). By use of the quadrature formula the corresponding solutions of the finite initial-value problems (53.2), (54) and (55.2), (56) are obtained. These solutions are used to yield a solution of the finite initial-value problem (49.2), (50).

Using the coefficients (48), define

$$A_n = -\frac{m_{nj}}{\rho_{n+1,j}} = -\frac{ma_n}{\rho_{n+1}}, \quad n\geq 0,$$

$$B_n = \frac{\rho_{nj} + \rho_{n+1,j}}{\rho_{n+1,j}} = 1 + \frac{\rho_n}{\rho_{n+1}}, \quad n\geq 0, \tag{57.1}$$

$$C_n = \frac{\rho_{nj}}{\rho_{n+1,j}} = \frac{\rho_n}{\rho_{n+1}}, \quad n\geq 1,$$

and

$$D_0 = -\frac{m_{i0}}{2\mu_{i1}} = -\frac{mb_0}{2\mu_1}, \quad D_n = -\frac{m_{in}}{\mu_{i,n+1}} = -\frac{mb_n}{\mu_{n+1}}, \quad n\geq 1,$$

$$E_0 = 1, \quad E_n = \frac{\mu_{in} + \mu_{i,n+1}}{\mu_{i,n+1}} = 1 + \frac{\mu_n}{\mu_{n+1}}, \quad n\geq 1, \tag{57.2}$$

$$F_n = \frac{\mu_{in}}{\mu_{i,n+1}} = \frac{\mu_n}{\mu_{n+1}}, \quad n\geq 1.$$

Let $\{P_n(x)\}$ $(n\geq 0)$, $\{R_n(y)\}$ $(n\geq 0)$ and $\{S_n(y)\}$ $(n\geq 0)$ be the sequences of polynomials generated, respectively, by the three-term recurrences

$$P_0 = 1$$

$$P_1(x) = A_0 x + B_0 \tag{58.1}$$

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad n\geq 1,$$

$$R_0 = 1$$

$$R_1 = D_0 y + E_0 \tag{58.2}$$

$$R_{n+1}(y) = (D_n y + E_n)R_n(y) - F_n R_{n-1}(y), \quad n \geq 1,$$

$$S_0 = 0, \; S_1 = 1$$

$$S_2(y) = D_1 y + E_1 \tag{58.3}$$

$$S_{n+1}(y) = (D_n y + E_n)S_n(y) - F_n S_{n-1}(y), \quad n \geq 2.$$

The coefficients of each of the three recurrences (58.1), (58.2) and (58.3) satisfy condition (2); hence associated with each sequence of polynomials $\{P_n\}$ $(n\geq 0)$, $\{R_n\}$ $(n\geq 0)$ and $\{S_n\}$ $(n\geq 0)$ there exists a normalized integrator and an interval of orthogonality. Denote the corresponding polynomial sequences, normalized integrators, and intervals of orthogonality by $\{P_n, \alpha, [a,b]\}$, $\{R_n, \omega_1, [c_1, d_1]\}$, $\{S_n, \omega_2, [c_2, d_2]\}$. Denote the zeros of $P_N(x)$ by $x_1, x_2, \ldots, x_N$, the zeros of $R_M(y)$ by $y_1, y_2, \ldots, y_M$, and the zeros of $S_M(y)$ by $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{M-1}$; and let $\{\lambda_i\}$, $1\leq i\leq N$, $\{\kappa_j\}$, $1\leq j\leq M$, and $\{\nu_j\}$, $1\leq j\leq M-1$, be the corresponding sets of Christoffel numbers.

Let

$$\gamma_n = \int_a^b P_n^2(x)\,d\alpha(x), \; n\geq 0,$$

$$\xi_n = \int_{c_1}^{d_1} R_n^2(y)\,d\omega_1(y),\ n\geq 0,$$

$$\zeta_0 = 1,\quad \zeta_n = \int_{c_2}^{d_2} S_n^2(y)\,d\omega_2(y),\ n\geq 1;$$

and for each pair of integers $i\geq 0$, $j\geq 0$ define

$$h_{ij}(x,y,t) = (1+\delta_{0q})\,\frac{a_{pq}}{2}\,\frac{P_p(x)}{\gamma_p}\,\frac{R_q(y)}{\xi_q}\,P_i(x)R_j(y)e^{-(x+y)t} \tag{59.1}$$

and

$$k_{ij}(x,y,t) = (1-\delta_{0q})\,\frac{a_{pq}}{2}\,\frac{P_p(x)}{\gamma_p}\,\frac{S_q(y)}{\zeta_q}\,P_i(x)S_j(y)e^{-(x+y)t}. \tag{59.2}$$

Theorem 7 may be applied to obtain a solution of the infinite initial-value problem (53.1), (54) and its finite truncation (53.2), (54). One finds that $\{W_{ij}\}$ given by

$$W_{ij}(t) = \int_a^b \int_{c_1}^{d_1} h_{ij}(x,y,t)\,d\omega_1(y)\,d\alpha(x),\ i\geq 0,\ j\geq 0,$$

is a solution of (53.1), (54); that $\{w_{ij}\}$ given by

$$w_{ij}(t) = \sum_{r=1}^{N}\sum_{s=1}^{M}\lambda_r\kappa_s h_{ij}(x_r,y_s,t),\quad 0\leq i\leq N-1,\ 0\leq j\leq M-1,$$

is a solution of (53.2), (54); and that for each $i=0,1,\ldots,N-1$, $j=0,1,\ldots,M-1$, and each $t\geq 0$ there exist $\hat{x}_1,\hat{x}_2\epsilon[a,b]$, $\hat{y}_1,\hat{y}_2\epsilon[c_1,d_1]$ such that

$$W_{ij}(t) = w_{ij} + E_w,$$

where

$$E_w = \frac{1}{[(2N)!]} \frac{C_1 C_2 \dots C_N}{A_0 A_1^2 \dots A_{N-1}^2 A_N} \frac{\partial^{2N} h_{ij}}{\partial x^{2N}} (\hat{x}_1, \hat{y}_1, t)$$

$$+ \frac{1}{[(2M)!]} \frac{F_1 F_2 \dots F_M}{D_0 D_1^2 \dots D_{M-1}^2 D_M} \frac{\partial^{2M} h_{ij}}{\partial y^{2M}} (\hat{x}_2, \hat{y}_2, t).$$

Note that the initial-value problem (55.1), (56) may also be solved by applying Theorem 7 if one shifts the second subscript j in the equations (55.1). One finds that $\{Z_{ij}\}$ given by

$$Z_{ij}(t) = \int_a^b \int_{c_2}^{d_2} k_{ij}(x,y,t) d\omega_2(y) d\alpha(x), \quad i \geq 0,\ j \geq 0,$$

is a solution of (55.1), (56); that $\{z_{ij}\}$ given by

$$z_{ij}(t) = \sum_{r=1}^{N} \sum_{s=1}^{M-1} \lambda_r \nu_s k_{ij}(x_r, \tilde{y}_s, t), \quad 0 \leq i \leq N-1,\ 0 \leq j \leq M-1,$$

is a solution of (55.1), (56); and that for each $i=0,1,\dots,N-1$, $j=0,1,\dots,M-1$, and each $t \geq 0$ there exist $x_1^*, x_2^* \epsilon [a,b]$, $y_1^*, y_2^* \epsilon [c_2, d_2]$ such that

$$Z_{ij}(t) = z_{ij}(t) + E_z,$$

where

$$E_z = \frac{1}{[(2N)!]} \frac{C_1 C_2 \cdots C_N}{A_0 A_1^2 \cdots A_{N-1}^2 A_N} \frac{\partial^{2N} k_{ij}}{\partial x^{2N}} (x_1^*, y_1^*, t)$$

$$+ \frac{1}{[(2M-2)!]} \frac{F_2 F_3 \cdots F_M}{D_1 D_2^2 \cdots D_{M-1}^2 D_M} \frac{\partial^{(2M-2)} k_{ij}}{\partial y^{(2M-2)}} (x_2^*, y_2^*, t).$$

Note that if $q=0$, then $k_{ij}(x,y,t) \equiv 0$ and hence $Z_{ij}(t) \equiv 0$ and $z_{ij}(t) \equiv 0$. Summarizing the preceding remarks and using equations (51) and (52) yields

Theorem 9. Suppose that the coefficients in (58.1), (58.2) and (58.3) satisfy condition (2)--that is, the P_n $(n\geq 0)$ given by (58.1) are orthogonal on $[a,b]$ with respect to $\alpha(x)$, the R_n $(n\geq 0)$ given by (58.2) are orthogonal on $[c_1,d_1]$ with respect to $\omega_1(y)$, and the S_n $(n\geq 1)$ given by (58.3) are orthogonal on $[c_2,d_2]$ with respect to $\omega_2(y)$. Then

[i] if $q = 0$:

(a) for $i\geq 0$, $-\infty<j<\infty$ the sequence $\{V_{ij}\}$ given by $V_{ij}(t) = W_{i|j|}(t)$ is a solution of the infinite initial-value problem (49.1), (50);

(b) for $0\leq i\leq N-1$, $-M+1\leq j\leq M-1$ the sequence $\{v_{ij}\}$ given by $v_{ij}(t) = w_{i|j|}(t)$ is a solution of the finite initial-value problem (49.2), (50);

(c) for each $0\leq i\leq N-1$, $-M+1\leq j\leq M-1$ and each $t\geq 0$

$$V_{ij}(t) = v_{ij}(t) + E_w;$$

[ii] if $q \neq 0$:

(a) for $i \geq 0$, $-\infty < j < \infty$ the sequence $\{V_{ij}\}$ given by

$$V_{i,-j}(t) = W_{ij}(t) - Z_{ij}(t), \quad i \geq 0,\ j > 0,$$

$$V_{ij}(t) = W_{ij}(t) + Z_{ij}(t), \quad i \geq 0,\ j \geq 0,$$

is a solution of the infinite initial-value problem (49.1), (50);

(b) for $0 \leq i \leq N-1$, $-M+1 \leq j \leq M-1$ the sequence $\{v_{ij}\}$ given by

$$v_{i,-j}(t) = w_{ij}(t) - z_{ij}(t), \quad i \geq 0,\ j > 0,$$

$$v_{ij}(t) = w_{ij}(t) + z_{ij}(t), \quad i \geq 0,\ j \geq 0,$$

is a solution of the finite initial-value problem (49.2), (50);

(c) for each $0 \leq i \leq N-1$, $-M+1 \leq j \leq M-1$ and each $t \geq 0$

$$V_{ij}(t) = v_{ij}(t) + E_w - E_z, \quad 0 \leq i \leq N-1,\ -M+1 \leq j \leq M-1,$$

$$V_{ij}(t) = v_{ij}(t) + E_w + E_z, \quad 0 \leq i \leq N-1,\ 0 \leq j \leq M-1.$$

Using the coefficients (48), one can construct a system of second-order differential equations corresponding to the linearized equations of motion of a half-planar array of coupled oscillators each subjected to viscous damping proportional to its mass. Such a system is

$$m_{i,-j}(\ddot{y}_{i,-j}+\beta\dot{y}_{i,-j}) = \mu_{i,-j}(y_{i,-j+1}-y_{i,-j}) + \mu_{i,-j-1}(y_{i,-j-1}-y_{i,-j})$$

$$+ \rho_{i,-j}(y_{i-1,-j}-y_{i,-j}) + \rho_{i+1,-j}(y_{i+1,-j}-y_{i,-j})$$

$$(i\geq 0,\ j\geq 1),$$

$$m_{i0}(\ddot{y}_{i0}+\beta\dot{y}_{i0}) = \mu_{i,-1}(y_{i,-1}-y_{i0}) + \mu_{i1}(y_{i1}-y_{i0}) \qquad (60.1)$$

$$+ \rho_{i0}(y_{i-1,0}-y_{i0}) + \rho_{i+1,0}(y_{i+1,0}-y_{i0}) \ (i\geq 0),$$

$$m_{ij}(\ddot{y}_{ij}+\beta\dot{y}_{ij}) = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij})$$

$$+ \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij}) \ (i\geq 0,\ j\geq 1)$$

with its finite truncation

$$m_{i,-j}(\ddot{y}_{i,-j}+\beta\dot{y}_{i,-j}) = \mu_{i,-j}(y_{i,-j+1}-y_{i,-j}) + \mu_{i,-j-1}(y_{i,-j-1}-y_{i,-j})$$

$$+ \rho_{i,-j}(y_{i-1,-j}-y_{i,-j}) + \rho_{i+1,-j}(y_{i+1,-j}-y_{i,-j})$$

$$(0\leq i\leq N-1,\ 1\leq j\leq M-1),$$

$$m_{i0}(\ddot{y}_{i0}+\beta\dot{y}_{i0}) = \mu_{i,-1}(y_{i,-1}-y_{i0}) + \mu_{i1}(y_{i1}-y_{i0}) \qquad (60.2)$$

$$+ \rho_{i0}(y_{i-1,0}-y_{i0}) + \rho_{i+1,0}(y_{i+1,0}-y_{i0})$$

$$(0 \le i \le N-1),$$

$$m_{ij}(\ddot{y}_{ij}+\beta\dot{y}_{ij}) = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij})$$

$$+ \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij})$$

$$(0 \le i \le N-1,\ 1 \le j \le M-1),$$

both subject to the initial conditions

$$y_{pq}(0) = a_{pq}, \quad \dot{y}_{pq}(0) = b_{pq},$$

$$y_{ij}(0) = 0, \quad y_{ij}(0) = 0, \quad i \ne p,\ j \ne q. \qquad (61)$$

It is clear that the procedure used to solve the infinite first-order initial-value problem (49.1), (50) and its finite truncation (49.2), (50) can be used to solve the infinite second-order initial-value problem (60.1), (61) and its finite truncation (60.2), (61). The modification required is in the definition of the expressions analogous to (59.1) and (59.2). One easily finds that the appropriate expressions are

$$H_{ij}(x,y,t) = (1+\delta_{0q})P_i(x)\,\frac{P_p(x)}{\gamma_p}\,R_j(y)\,\frac{R_q(y)}{\xi_q}\,K(x,y,t), \qquad (62.1)$$

$$i\geq 0,\ j\geq 0,$$

and

$$K_{ij}(x,y,t) = (1-\delta_{0q})P_i(x)\,\frac{P_p(x)}{\gamma_p}\,S_j(y)\,\frac{S_q(y)}{\zeta_q}\,K(x,y,t), \qquad (62.2)$$

$$i\geq 0,\ j\geq 0,$$

where

$$K(x,y,t) = e^{-\frac{\beta t}{2}}\left\{a_{pq}\cos\sqrt{x+y-\frac{\beta^2}{4}}\,t\right.$$

$$\left. + \left\{\frac{a_{pq}\beta}{2} + b_{pq}\right\}\frac{\sin\sqrt{x+y-\frac{\beta^2}{4}}\,t}{\sqrt{x+y-\frac{\beta^2}{4}}}\right\}.$$

<u>Theorem 10</u>. Suppose that the coefficients in (58.1), (58.2) and (58.3) satisfy condition (2)--that is, the P_n $(n\geq 0)$ given by (58.1) are orthogonal on $[a,b]$ with respect to $\alpha(x)$, the R_n $(n\geq 0)$ given by (58.2) are orthogonal on $[c_1,d_1]$ with respect to $\omega_1(y)$, and the S_n $(n\geq 1)$ given by (58.3) are orthogonal on $[c_2,d_2]$ with respect to $\omega_2(y)$. For $q = 0$ and any $i\geq 0$, $-\infty<j<\infty$ define

$$X_{ij}(t) = \int_a^b \int_{c_1}^{d_1} H_{i|j|}(x,y,t)d\omega_1(y)d\alpha(x) \tag{63.1}$$

and

$$x_{ij} = \sum_{r=1}^{N} \sum_{s=1}^{M} \lambda_r \kappa_s H_{i|j|}(x_r,y_s,t). \tag{63.2}$$

For $q \neq 0$ and any $i \geq 0$, $-\infty < j < \infty$ define

$$X_{ij}(t) = \int_a^b \int_{c_1}^{d_1} H_{i,-j}(x,y,t)d\omega_1(y)d\alpha(x) \tag{64.1}$$

$$- \int_a^b \int_{c_2}^{d_2} K_{i,-j}(x,y,t)d\omega_2(y)d\alpha(x), \quad i \geq 0,\ j \leq -1,$$

$$X_{ij}(t) = \int_a^b \int_{c_1}^{d_1} H_{ij}(x,y,t)d\omega_1(y)d\alpha(x)$$

$$+ \int_a^b \int_{c_2}^{d_2} K_{ij}(x,y,t)d\omega_2(y)d\alpha(x), \quad i \geq 0,\ j \geq 0,$$

and

$$x_{ij}(t) = \sum_{r=1}^{N} \sum_{s=1}^{M} \lambda_r \kappa_s H_{i,-j}(x_r,y_s,t) \tag{64.2}$$

$$- \sum_{r-1}^{N} \sum_{s=1}^{M-1} \lambda_r \nu_s K_{i,-j}(x_r,\tilde{y}_s,t), \quad i \geq 0,\ j \leq -1,$$

$$x_{ij}(t) = \sum_{r=1}^{N} \sum_{s=1}^{M} \lambda_r \kappa_s H_{ij}(x_r, y_s, t) + \sum_{r=1}^{N} \sum_{s=1}^{M-1} \lambda_r \nu_s K_{ij}(x_r, \tilde{y}_s, t),$$

$$i \geq 0,\ j \geq 0.$$

Suppose that $\dot{X}_{ij}(t)$ and $\ddot{X}_{ij}$ are given by differentiation with respect to t under the integral sign.

Then:

(a) for $i \geq 0$, $-\infty < j < \infty$ the sequence $\{X_{ij}\}$ is a solution of the infinite initial-value problem (60.1), (61);

(b) for $0 \leq i \leq N-1$, $-M+1 \leq j \leq M-1$ the sequence $\{x_{ij}\}$ is a solution of the finite initial-value problem (60.2), (61);

(c) for each $0 \leq i \leq N-1$, $-M+1 \leq j \leq M-1$ and each $t \geq 0$ there exist $\hat{x}_1, \hat{x}_2, x_1^*, x_2^* \epsilon [a,b]$ and $\hat{y}_1, \hat{y}_2 \epsilon [c_1, d_1]$, $y_1^*, y_2^* \epsilon [c_2, d_2]$ such that

$$X_{ij}(t) = x_{ij}(t) + E_H - E_K, \quad i \geq 0,\ j \leq -1,$$

$$X_{ij}(t) = x_{ij}(t) + E_H + E_K, \quad i \geq 0,\ j \geq 0,$$

where

$$E_H = \frac{1}{[(2N)!]} \frac{C_1 C_2 \ldots C_N}{A_0 A_1^2 \ldots A_{N-1}^2 A_N} \frac{\partial^{2N} H_{i|j|}}{\partial x^{2N}} (\hat{x}_1, \hat{y}_1, t)$$

$$+ \frac{1}{[(2M)!]} \frac{F_1 F_2 \ldots F_M}{D_0 D_1^2 \ldots D_{M-1}^2 D_M} \frac{\partial^{2M} H_{i|j|}}{\partial y^{2M}} (\hat{x}_2, \hat{y}_2, t),$$

$$E_K = \frac{1}{[(2N)!]}\,\frac{C_1C_2\ldots C_N}{A_0A_1^2\ldots A_{N-1}^2A_N}\,\frac{\partial^{2M}K_{i|j|}}{\partial x^{2N}}\,(x_1^*,y_1^*,t)$$

$$\frac{1}{[(2M-2)!]}\,\frac{F_2F_3\ldots F_M}{D_1D_2^2\ldots D_{M-1}^2D_M}\,\frac{\partial^{(2M-2)}K_{i|j|}}{\partial y^{(2M-2)}}\,(x_2^*,y_2^*,t).$$

One might note that the rather lengthy expressions used to give solutions in Theorem 10 may be written in the following more compact, though less lucid, form

$$X_{ij}(t) = \int_a^b \int_{c_1}^{d_1} H_{i|j|}(x,y,t)d\omega_1(y)d\alpha(x)$$

$$+ \operatorname{sgn}(jq) \int_a^b \int_{c_2}^{d_2} K_{i|j|}(x,y,t)d\omega_2(y)d\alpha(x),$$

where $\operatorname{sgn}(0) \stackrel{\Delta}{=} 0$.

This form has the advantage of obviating the supposition that q be non-negative, which is a convenience if one wishes to superpose two solutions for which q has opposite signs. Because of the physical symmetry of the array, the supposition that $q \geq 0$ can always be satisfied by suitably matching the nomenclature to the array so long as non-zero initial conditions are specified on only one inertial element.

Planar Arrays

By using the previously described decomposition technique and separation procedure, one may also obtain solutions of first- and second-order infinite initial-value problems corresponding to arrays which cover the entire plane and have physical symmetry about both a row and a column of inertial elements. This procedure will be described in detail for first-order systems and the analogous result simply stated for second-order systems. The signum notation introduced at the end of the preceding section will be used here for the sake of economy.

Let $\{\rho_i\}$ $(i\geq 1)$ and $\{\mu_j\}$ $(j\geq 1)$ be two positive sequences; let $\{a_i\}$ $(i\geq 0)$ and $\{b_j\}$ $(j\geq 0)$ be two positive sequences and m a positive constant. Define

$$
\begin{aligned}
m_{ij} &= m a_{|i|} b_{|j|}, && -\infty<i<\infty,\ -\infty<j<\infty,\\
\rho_{ij} &= \rho_{|i|} b_{|j|}, && -\infty<i<\infty,\ -\infty<j<\infty,\ i\neq 0, \qquad (65)\\
\mu_{ij} &= a_{|i|} \mu_{|j|}, && -\infty<i<\infty,\ -\infty<j<\infty,\ j\neq 0.
\end{aligned}
$$

Let p and q be integers, and let a_{pq} be a given constant. Using the coefficients (65), construct the first-order infinite system

$$
\begin{aligned}
m_{-i,-j}\dot{y}_{-i,-j} &= \mu_{-i,-j}(y_{-i,-j+1}-y_{-i,-j}) + \mu_{-i,-j-1}(y_{-i,-j-1}-y_{-i,-j})\\
&\quad + \rho_{-i,-j}(y_{-i+1,-j}-y_{-i,-j}) + \rho_{-i-1,-j}(y_{-i-1,-j}-y_{-i,-j}),\\
&\qquad i\geq 1,\ j\geq 1,
\end{aligned}
$$

$$m_{i,-j}\dot{y}_{i,-j} = \mu_{i,-j}(y_{i,-j+1}-y_{i,-j}) + \mu_{i,-j-1}(y_{i,-j-1}-y_{i,-j})$$

$$+ \rho_{i,-j}(y_{i-1,-j}-y_{i,-j}) + \rho_{i+1,-j}(y_{i+1,-j}-y_{i,-j}),$$

$$i \geq 1,\ j \geq 1,$$

$$m_{-ij}\dot{y}_{-ij} = \mu_{-ij}(y_{-i,j-1}-y_{-ij}) + \mu_{-i,j+1}(y_{-i,j+1}-y_{-ij})$$

$$+ \rho_{-ij}(y_{-i+1,j}-y_{-ij}) + \rho_{-i-1,j}(y_{-i-1,j}-y_{-ij}),$$

$$i \geq 1,\ j \geq 1,$$

$$m_{ij}\dot{y}_{ij} = \mu_{ij}(y_{i,j-1}-y_{ij}) + \mu_{i,j+1}(y_{i,j+1}-y_{ij}) \tag{66}$$

$$+ \rho_{ij}(y_{i-1,j}-y_{ij}) + \rho_{i+1,j}(y_{i+1,j}-y_{ij}),$$

$$i \geq 1,\ j \geq 1,$$

$$m_{i0}\dot{y}_{i0} = \mu_{i,-1}(y_{i,-1}-y_{i0}) + \mu_{i1}(y_{i1}-y_{i0})$$

$$+ \rho_{i0}(y_{i-1,0}-y_{i0}) + \rho_{i+1,0}(y_{i+1,0}-y_{i0}),\quad i \geq 1,$$

$$m_{-i0}\dot{y}_{-i0} = \mu_{-i,-1}(y_{-i,-1}-y_{-i0}) + \mu_{-i1}(y_{-i1}-y_{-i0})$$

$$+ \rho_{-i0}(y_{-i+1,0}-y_{-i0}) + \rho_{-i-1,0}(y_{-i-1,0}-y_{-i0}), \quad i\geq 1,$$

$$m_{0j}\dot{y}_{0j} = \mu_{0j}(y_{0,j-1}-y_{0j}) + \mu_{0,j+1}(y_{0,j+1}-y_{0j})$$

$$+ \rho_{-1j}(y_{-1j}-y_{0j}) + \rho_{1j}(y_{1j}-y_{0j}), \quad j\geq 1,$$

$$m_{0,-j}\dot{y}_{0,-j} = \mu_{0,-j}(y_{0,-j+1}-y_{0,-j}) + \mu_{0,-j-1}(y_{0,-j-1}-y_{0,-j})$$

$$+ \rho_{-1,-j}(y_{-1,-j}-y_{0,-j}) + \rho_{1,-j}(y_{1,-j}-y_{0,-j}), \quad j\geq 1,$$

$$m_{00}\dot{y}_{00} = \mu_{0,-1}(y_{0,-1}-y_{00}) + \mu_{01}(y_{01}-y_{00})$$

$$+ \rho_{-10}(y_{-10}-y_{00}) + \rho_{10}(y_{10}-y_{00})$$

subject to the initial conditions

$$y_{pq}(0) = a_{pq} \tag{67}$$

$$y_{ij}(0) = 0, \quad i\neq p,\ j\neq q.$$

The differential equations (66) may be interpreted as the equations of motion, in terms of velocity, of an infinite array of masses each coupled to its nearest neighbor by viscous friction. The y_{ij} are to be interpreted as velocities normal to the plane of the array (see Figure 7).

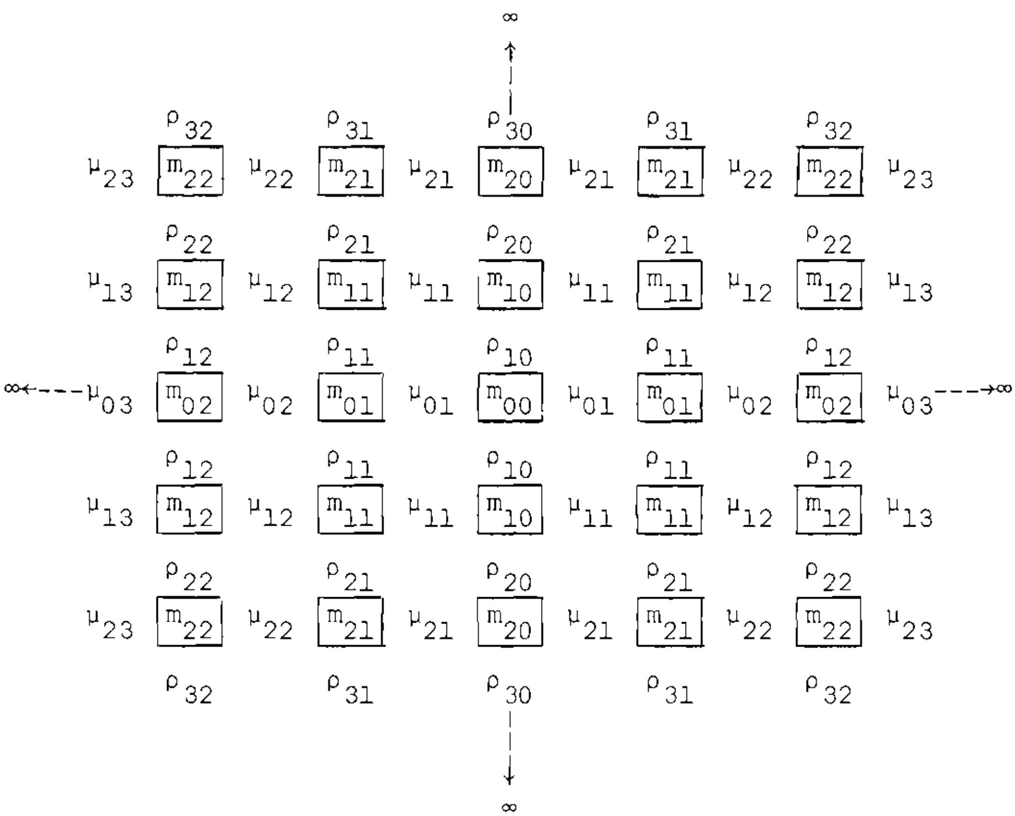


Figure 7. An Infinite Planar Array of Sliding Plates

A solution of the initial-value problem will be obtained by decomposing the differential equations and initial conditions into four equivalent systems corresponding to quarter-planar arrays. Given any sequence of functions $\{y_{ij}\}$, $-\infty<i<\infty$, $-\infty<j<\infty$, define the four sequences of functions $\{w_{ij}\}$, $\{z_{ij}\}$, $\{u_{ij}\}$ and $\{v_{ij}\}$, $i\geq 0$, $j\geq 0$, by

$$w_{ij} = \frac{y_{ij} + y_{-ij} + y_{i,-j} + y_{-i,-j}}{4}, \quad i\geq 0,\ j\geq 0,$$

$$z_{ij} = \frac{y_{ij} + y_{-ij} - y_{i,-j} - y_{-i,-j}}{4}, \quad i\geq 0,\ j\geq 0,$$

$$u_{ij} = \frac{y_{ij} - y_{-ij} + y_{i,-j} - y_{-i,-j}}{4}, \quad i\geq 0,\ j\geq 0, \tag{68}$$

$$v_{ij} = \frac{y_{ij} - y_{-ij} - y_{i,-j} + y_{-i,-j}}{4}, \quad i\geq 0,\ j\geq 0.$$

Note that from (68) it follows that

$$y_{ij} = w_{|i||j|} + \mathrm{sgn}(j)z_{|i||j|} + \mathrm{sgn}(i)u_{|i||j|} + \mathrm{sgn}(ij)v_{|i||j|}; \tag{69}$$

that $z_{i0} \equiv 0$ and $v_{i0} \equiv 0$, $i\geq 0$; and that $u_{0j} \equiv 0$ and $v_{0j} \equiv 0$, $j\geq 0$. The differential equations (66) may be expressed entirely in terms of w_{ij}, z_{ij}, u_{ij}, v_{ij} and then combined to obtain the following four systems of differential equations and initial conditions which are equivalent to (66), (67). The $\{w_{ij}\}$ ($i\geq 0$, $j\geq 0$) must satisfy the differential equations

$$m_{00}\dot{w}_{00} = 2\mu_{01}(w_{01}-w_{00}) + 2\rho_{10}(w_{10}-w_{00}),$$

$$m_{0j}\dot{w}_{0j} = \mu_{0j}(w_{0,j-1}-w_{0j}) + \mu_{0,j+1}(w_{0,j+1}-w_{0j}) \tag{70}$$

$$+ 2\rho_{1j}(w_{1j}-w_{0j}), \quad j\geq 1,$$

$$m_{i0}\dot{w}_{i0} = 2\mu_{i1}(w_{i1}-w_{i0}) + \rho_{i0}(w_{i-1,0}-w_{i0}) + \rho_{i+1,0}(w_{i+1,0}-w_{i0}), \quad i\geq 1,$$

$$m_{ij}\dot{w}_{ij} = \mu_{ij}(w_{i,j-1}-w_{ij}) + \mu_{i,j+1}(w_{i,j+1}-w_{ij})$$

$$+ \rho_{ij}(w_{i-1,j}-w_{ij}) + \rho_{i+1,j}(w_{i+1,j}-w_{ij}), \quad i\geq 1,\ j\geq 1,$$

subject to the initial conditions

$$w_{|p||q|}(0) = (1+\delta_{0p})(1+\delta_{0q})\,\frac{a_{pq}}{4}, \tag{70.1}$$

$$w_{ij}(0) = 0, \quad i\neq|p|,\ j\neq|q|.$$

Similarly the $\{z_{ij}\}$ ($i\geq 0$, $j\geq 0$) must satisfy the differential equations

$$z_{i0} \equiv 0, \quad i\geq 0,$$

$$m_{0j}\dot{z}_{0j} = \mu_{0j}(z_{0,j-1}-z_{0j}) + \mu_{0,j+1}(z_{0,j+1}-z_{0j}) \tag{71}$$

$$+ 2\rho_{1j}(z_{1j}-z_{0j}), \quad j\geq 1,$$

$$m_{ij}\dot{z}_{ij} = \mu_{ij}(z_{i,j-1}-z_{ij}) + \mu_{i,j+1}(z_{i,j+1}-z_{ij})$$

$$+ \rho_{ij}(z_{i-1,j}-z_{ij}) + \rho_{i+1,j}(z_{i+1,j}-z_{ij}), \quad i\geq 1,\ j\geq 1,$$

subject to the initial conditions

$$z_{|p||q|}(0) = (1+\delta_{0p})\mathrm{sgn}(q)\,\frac{a_{pq}}{4}\,,$$

$$z_{ij}(0) = 0, \quad i\neq|p|,\ j\neq|q|; \tag{71.1}$$

the $\{u_{ij}\}$ must satisfy the differential equations

$$u_{0j} \equiv 0, \quad j\geq 0,$$

$$m_{i0}\dot{u}_{i0} = 2\mu_{i1}(u_{i1}-u_{i0}) + \rho_{i0}(u_{i-1,0}-u_{i0}) \tag{72}$$

$$+ \rho_{i+1,0}(u_{i+1,0}-u_{i0}), \quad i\geq 1,$$

$$m_{ij}\dot{u}_{i0} = \mu_{ij}(u_{i,j-1}-u_{ij}) + \mu_{i,j+1}(u_{i,j+1}-u_{ij})$$

$$+ \rho_{ij}(u_{i-1,j}-u_{ij}) + \rho_{i+1,j}(u_{i+1,j}-u_{ij}), \quad i\geq 0,\ j\geq 0,$$

subject to the initial conditions

$$u_{|p||q|}(0) = (1+\delta_{0q})\mathrm{sgn}(p)\,\frac{a_{pq}}{4}\,,$$

$$u_{ij}(0) = 0, \quad i\neq|p|,\ j\neq|q|; \tag{72.1}$$

and the $\{v_{ij}\}$ must satisfy the differential equations

$$v_{i0} \equiv 0,\ i\geq 0, \quad v_{0j} \equiv 0,\ j\geq 0, \tag{73}$$

$$m_{ij}\dot{v}_{ij} = \mu_{ij}(v_{i,j-1}-v_{ij}) + \mu_{i,j+1}(v_{i,j+1}-v_{ij}) \tag{73}$$

$$+ \rho_{ij}(v_{i-1,j}-v_{ij}) + \rho_{i+1,j}(v_{i+1,j}-v_{ij}), \quad i\geq 1,\ j\geq 1,$$

subject to the initial conditions

$$\begin{aligned} v_{|p||q|}(0) &= \operatorname{sgn}(pq)\,\frac{a_{pq}}{4}\,, \\ v_{ij}(0) &= 0, \quad i\neq|p|,\ j\neq|q|. \end{aligned} \tag{73.1}$$

The coefficients in the differential equations (70), (71), (72) and (73) are used to generate four sequences of orthogonal polynomials which are subsequently used to construct solutions of the four initial-value problems (70), (70.1) through (73), (73.1).

Define

$$\begin{aligned} A_0 &= -\frac{m_{0j}}{2\rho_{1j}} = -\frac{ma_0}{2\rho_1}\,, \quad A_n = -\frac{m_{nj}}{\rho_{n+1,j}} = -\frac{ma_n}{\rho_{n+1}}\,, \quad n\geq 1, \\ B_0 &= 1, \quad B_n = \frac{\rho_{nj}+\rho_{n+1,j}}{\rho_{n+1,j}} = 1 + \frac{\rho_n}{\rho_{n+1}}\,, \quad n\geq 1, \\ C_n &= \frac{\rho_{nj}}{\rho_{n+1,j}} = \frac{\rho_n}{\rho_{n+1}}\,, \quad n\geq 1, \end{aligned} \tag{74.1}$$

and

$$D_0 = -\frac{m_{i0}}{2\mu_{i1}} = -\frac{mb_0}{2\mu_1}\,; \quad D_n = -\frac{m_{in}}{\mu_{i,n+1}} = -\frac{mb_n}{2\mu_{n+1}}\,, \quad n\geq 1,$$

$$E_0 = 1, \quad E_n = \frac{\mu_{in} + \mu_{i,n+1}}{\mu_{i,n+1}} = 1 + \frac{\mu_n}{\mu_{n+1}}, \quad n\geq 1, \tag{74.2}$$

$$F_n = \frac{\mu_{i,n}}{\mu_{i,n+1}} = \frac{\mu_n}{\mu_{n+1}}, \quad n\geq 1.$$

Let $\{P_n(x)\}$ $(n\geq 0)$, $\{Q_n(x)\}$ $(n\geq 0)$, $\{R_n(y)\}$ $(n\geq 0)$ and $\{S_n(y)\}$ $(n\geq 0)$ be the sequences of polynomials generated by the four three-term recurrences

$$\begin{aligned} &P_0(x) = 1 \\ &P_1(x) = A_0x + B_0 \\ &P_{n+1}(x) = (A_nx+B_n)P_n(x) - C_nP_{n-1}(x), \quad n\geq 1, \end{aligned} \tag{75.1}$$

$$\begin{aligned} &Q_0 = 0, \quad Q_1 = 1 \\ &Q_2(x) = A_1x + B_1 \\ &Q_{n+1}(x) = (A_nx+B_n)Q_n(x) - C_nQ_{n-1}(x), \quad n\geq 2, \end{aligned} \tag{75.2}$$

$$\begin{aligned} &R_0 = 1 \\ &R_1(y) = D_0y + E_0 \end{aligned} \tag{75.3}$$

$$R_{n+1}(y) = (D_n y+E_n)R_n(y) - F_n R_{n-1}(y), \quad n\geq 1,$$

and

$$S_0 = 0, \quad S_1 = 1$$

$$S_2(y) = D_1 y + E_1 \tag{75.4}$$

$$S_{n+1}(y) = (D_n y+E_n)S_n(y) - F_n S_{n-1}(y), \quad n\geq 2.$$

The coefficients of each of the recurrences (75.1) through (75.4) satisfy condition (2); hence associated with each recurrence is a normalized integrator and an interval of orthogonality. Denote the respective polynomial sequences, normalized integrators, and intervals by $\{P_n(x),\alpha_1(x),[a_1,b_1]\}$, $\{Q_n(x),\alpha_2(x),[a_2,b_2]\}$, $\{R_n(y),\omega_1(y),[c_1,d_1]\}$ and $\{S_n(y),\omega_2(y),[c_2,d_2]\}$.

Lemma 9. The systems of differential equations (70), (71), (72), (73) have solutions given, respectively, by

$$w_{ij}(t) = P_i(x)R_j(y)u(x,y,t), \quad i\geq 0,\ j\geq 0,$$

$$z_{ij}(t) = P_i(x)S_j(y)u(x,y,t), \quad i\geq 0,\ j\geq 0,$$

$$u_{ij}(t) = Q_i(x)R_j(y)u(x,y,t), \quad i\geq 0,\ j\geq 0,$$

$$v_{ij}(t) = Q_i(x)S_j(y)u(x,y,t), \quad i\geq 0,\ j\geq 0,$$

if and only if

$$\frac{\partial u}{\partial t} + (x+y)u = 0.$$

Proof. By substituting the assumed form for w_{ij} into the differential equations (70) and using recurrences (75.1) and (75.3), one finds that the differential equations (70) reduce to $ma_ib_jP_i(x)R_j(y)\left\{\frac{\partial u}{\partial t} + (x+y)u\right\} = 0$. Since this equality must hold for all $i\geq 0$, $j\geq 0$ and all x,y, the conclusion follows at once. The conclusions regarding z_{ij}, u_{ij} and v_{ij} may be shown similarly.

Let

$$\gamma_n = \int_{a_1}^{b_1} P_n^2(x)d\alpha_1(x), \quad n\geq 0,$$

$$\eta_0 = 1, \quad \eta_n = \int_{a_2}^{b_2} Q_n^2(x)d\alpha_2(x), \quad n\geq 1,$$

$$\xi_n = \int_{c_1}^{d_1} R_n^2(y)d\omega_1(y), \quad n\geq 0,$$

$$\zeta_0 = 1, \quad \zeta_n = \int_{c_2}^{d_2} S_n^2(y)d\omega_2(y), \quad n\geq 1,$$

and for each $i\geq 0$, $j\geq 0$ define

$$f_{ij}(x,y,t) = (1+\delta_{0p})(1+\delta_{0q})P_i(x)\,\frac{P_{|p|}(x)}{\gamma_{|p|}}\,R_j(y)\,\frac{R_{|q|}(y)}{\xi_{|q|}}\,u(x,y,t),$$

$$g_{ij}(x,y,t) = (1+\delta_{0p})P_i(x)\,\frac{P_{|p|}(x)}{\gamma_{|p|}}\,S_j(y)\,\frac{S_{|q|}(y)}{\zeta_{|q|}}\,u(x,y,t),$$

$$h_{ij}(x,y,t) = (1+\delta_{0q})Q_i(x)\,\frac{Q_{|p|}(x)}{\eta_{|p|}}\,R_j(y)\,\frac{R_{|q|}(y)}{\xi_{|q|}}\,u(x,y,t),$$

$$k_{ij}(x,y,t) = Q_i(x)\,\frac{Q_{|p|}(x)}{\eta_{|p|}}\,S_j(y)\,\frac{S_{|q|}(y)}{\zeta_{|q|}}\,u(x,y,t),$$

where $u(x,y,t) = \frac{a_{pq}}{4}\,e^{-(x+y)t}$.

<u>Lemma 10</u>. For each $i\geq 0$, $j\geq 0$ define

$$w_{ij}(t) = \int_{a_1}^{b_1}\int_{c_1}^{d_1} f_{ij}(x,y,t)\,d\omega_1(y)\,d\alpha_1(x),$$

$$z_{ij}(t) = \operatorname{sgn}(q)\int_{a_1}^{b_1}\int_{c_2}^{d_2} g_{ij}(x,y,t)\,d\omega_2(y)\,d\alpha_1(x),$$

$$u_{ij}(t) = \operatorname{sgn}(p)\int_{a_2}^{b_2}\int_{c_1}^{d_1} h_{ij}(x,y,t)\,d\omega_1(y)\,d\alpha_2(x),$$

$$v_{ij}(t) = \operatorname{sgn}(pq)\int_{a_2}^{b_2}\int_{c_2}^{d_2} k_{ij}(x,y,t)\,d\omega_2(y)\,d\alpha_2(x).$$

Suppose that $\dot{w}_{ij}$, $\dot{z}_{ij}$, $\dot{u}_{ij}$ and $\dot{v}_{ij}$ are each given by differentiation with respect to t under the integral.

Then:

(a) $\{w_{ij}\}$ is a solution of the initial-value problem (70), (70.1);

(b) $\{z_{ij}\}$ is a solution of the initial-value problem (71), (71.1);

(c) $\{u_{ij}\}$ is a solution of the initial-value problem (72), (72.1);

(d) $\{v_{ij}\}$ is a solution of the initial-value problem (73), (73.1).

Proof. Lemma 9 and the linearity of the integral imply that all the differential equations are satisfied. By using the orthogonality properties of the polynomial sequences, a direct calculation shows that the appropriate initial conditions are satisfied; e.g.,

$$v_{ij}(0) = \operatorname{sgn}(pq) \int_{a_2}^{b_2} \int_{c_2}^{d_2} Q_i(x) \frac{Q_{|p|}(x)}{\eta_{|p|}} S_j(y) \frac{S_{|q|}(y)}{\zeta_{|q|}} \frac{a_{pq}}{4} \, d\omega_2(y) d\alpha_2(x)$$

$$= \operatorname{sgn}(pq) \frac{a_{pq}}{4} \delta_{i|p|} \delta_{j|q|}.$$

The result of Lemma 10 may be combined with equation (69) to yield

Theorem 11. For all $-\infty<i<\infty$, $-\infty<j<\infty$ the sequence $\{y_{ij}\}$ defined by

$$\begin{aligned} y_{ij}(t) = & \int_{a_1}^{b_1} \int_{c_1}^{d_1} f_{|i||j|}(x,y,t) d\omega_1(y) d\alpha_1(x) \\ & + \operatorname{sgn}(jq) \int_{a_1}^{b_1} \int_{c_2}^{d_2} g_{|i||j|}(x,y,t) d\omega_2(y) d\alpha_1(x) \\ & + \operatorname{sgn}(ip) \int_{a_2}^{b_2} \int_{c_1}^{d_1} h_{|i||j|}(x,y,t) d\omega_1(y) d\alpha_2(x) \end{aligned}$$

$$+ \operatorname{sgn}(ijpq) \int_{a_2}^{b_2} \int_{c_2}^{d_2} k_{|i||j|}(x,y,t)\,d\omega_2(y)\,d\alpha_2(x)$$

is a solution of the infinite initial-value problem (66), (67).

Upon replacing the expressions $m_{ij}\dot{y}_{ij}$ in equations (66) by $m_{ij}(\ddot{y}_{ij}+\beta\dot{y}_{ij})$ and the non-zero initial conditions (67) by $y_{pq}(0) = a_{pq}$, $\dot{y}_{pq}(0) = b_{pq}$, one obtains a second-order system associated with an infinite planar array of damped oscillators. To obtain a solution of this second-order system one need only replace $u(x,y,t)$, which appears in the expressions f_{ij}, g_{ij}, h_{ij}, k_{ij} (cf. pp.77ff.), by

$$v(x,y,t) = \frac{e^{-\frac{\beta t}{2}}}{4}\left\{a_{pq}\cos\sqrt{x+y-\frac{\beta^2}{4}}\,t + \left(\frac{a_{pq}\beta}{2} + b_{pq}\right)\frac{\sin\sqrt{x+y-\frac{\beta^2}{4}}\,t}{\sqrt{x+y-\frac{\beta^2}{4}}}\right\}$$

and the solution $\{y_{ij}\}$ is given by Theorem 11.

CHAPTER IV

GENERALIZED WEIGHT FUNCTIONS FOR THE ORTHOGONAL POLYNOMIALS GENERATED BY SOME PERTURBED THREE-TERM RECURRENCE RELATIONS

The investigation in this chapter deals with two sequences of polynomials which arise in the solution of the equations of motion associated with infinite linear chains with an isotopic impurity. The first sequence of polynomials $\{M_n^{(\alpha,\beta)}\}$ considered is generated by a three-term recurrence relation which depends on two parameters. It will be shown that for certain values of the parameters the normalized integrator is not absolutely continuous (hence there exists no weight function) but is a Stieltjes integrator with a jump discontinuity. For $\alpha \neq 1$ the polynomials $\{M_n^{(\alpha,\beta)}\}$ exhibit some of the same properties as the Zolotareff elliptic polynomials described by Achyéser [1]. The procedure used to determine the interval of orthogonality and corresponding distribution (or weight function) has proved useful in several investigations of impurity problems. The second sequence of polynomials $\{A_n^{(\alpha)}\}$ considered is generated by a three-term recurrence in which the coefficients depend on the parameter α. It is shown that for all $\alpha > 0$ there always exists a weight function and that the interval of orthogonality depends on the value of α. For $\alpha = 1$ the polynomials $\{A_n^{(1)}\}$ are classical polynomials.

The Non-Classical Polynomials $M_n^{(\alpha,\beta)}$

Let $\alpha>0$ and $\beta>0$ be real numbers, and let $M_n^{(\alpha,\beta)}$ $(n\geq 0)$ be the polynomials given by

$$
\begin{aligned}
M_0^{(\alpha,\beta)}(x) &= 1 \\
M_1^{(\alpha,\beta)}(x) &= -\frac{\beta x}{2} + 1 \\
M_{2n}^{(\alpha,\beta)}(x) &= (-\alpha x+2)M_{2n-1}^{(\alpha,\beta)}(x) - M_{2n-2}^{(\alpha,\beta)}(x), \quad n\geq 1, \\
M_{2n+1}^{(\alpha,\beta)}(x) &= (-x+2)M_{2n}^{(\alpha,\beta)}(x) - M_{2n-1}^{(\alpha,\beta)}(x), \quad n\geq 1.
\end{aligned}
\tag{76}
$$

The only values of α and β for which the polynomials $M_n^{(\alpha,\beta)}$ are classical polynomials are $\alpha=1$ and either $\beta=1$ or $\beta=2$. To establish this result let

$$\Delta = (2b_2-b_1-b_0)[(b_1-b_0)^2+4(c_1+c_2)] + 9c_2(b_0-b_2),$$

$$g_1(n) = [(n+1)b_{n+1}+(1-n)b_n-b_1-b_0][(b_1-b_0)^2+4(c_1+c_2)]/3c_2$$

$$+ [(-2n-1)b_{n+1}+(2n-3)b_n+b_1+3b_0], \quad n\geq 1,$$

and

$$g_2(n) = [(n+1)b_nb_{n+1} - nb_n^2 - b_0b_1 + c_1 - (2n+1)c_{n+1} + (2n-3)c_n] \cdot$$

$$[(b_1-b_0)^2 + 4(c_1+c_2)]/3c_2$$

$$+ [(-2n-1)b_n b_{n+1} + (2n-1)b_n^2 + b_0 b_1 + b_0^2 + 4nc_{n+1}$$

$$+ (-4n+8)c_n], \quad n\geq 1,$$

where $b_0 = -\frac{2}{\beta}$, $b_{2n} = -2$ $(n\geq 1)$, $b_{2n+1} = -\frac{2}{\alpha}$ $(n\geq 1)$, $c_1 = \frac{2}{\alpha\beta}$, $c_n = \frac{1}{\alpha}$ $(n\geq 2)$ are the recurrence coefficients obtained by expressing (76) in monic form. A result due to Jayne [12] shows that the polynomials $M_n^{(\alpha,\beta)}$ are classical if and only if $\Delta = 0$, $g_1(n) = 0$ and $g_2(n) = 0$ for each $n\geq 1$. A direct calculation shows that

$$g_1(2n) = \frac{2}{3\alpha^2\beta^3}[2n\beta(\alpha-1)(4\alpha^2-2\alpha\beta^2+4\beta^2) + \alpha(1-\beta)(4\alpha^2-5\alpha\beta^2+4\beta^2)],$$

$$n\geq 1.$$

Hence a necessary condition that $g_1(2n) = 0$, $n\geq 1$, is that

$$\beta(\alpha-1)(4\alpha^2-2\alpha\beta^2+4\beta^2) = 0 \quad \text{and} \quad \alpha(1-\beta)(4\alpha^2-5\alpha\beta^2+4\beta^2) = 0.$$

From the first equation either $4\alpha^2 - 2\alpha\beta^2 + 4\beta^2 = 0$ or $\alpha = 1$. If $4\alpha^2 - 2\alpha\beta^2 + 4\beta^2 = 0$ the second equation becomes $\alpha(1-\beta)(-3\alpha\beta^2) = 0$, and hence $\beta = 1$; but then there exists no real α such that $4\alpha^2 - 2\alpha + 4 = 0$. Thus $\alpha = 1$ is necessary for $g_1(2n) = 0$, $n\geq 1$. With $\alpha = 1$ the second equation reduces to $(1-\beta)(4-\beta^2) = (1-\beta)(2-\beta)(2+\beta) = 0$. Consequently one finds that $\alpha = 1$ and either $\beta = 1$ or $\beta = 2$ are necessary conditions for $M_n^{(\alpha,\beta)}$ to be classical polynomials. A direct computation of Δ, $g_1(n)$ and $g_2(n)$ with $\alpha = 1$ and $\beta = 1$ or $\beta = 2$ shows that the polynomials

$M_n^{(1,1)}$ and $M_n^{(1,2)}$ are classical polynomials.

The polynomials $M_n^{(\alpha,\beta)}$ can be most conveniently represented in trigonometric form for all $\alpha>0$, $\beta>0$. Let Γ be the contour in the complex plane consisting of C_1 from ∞i to 0, C_2 along the real axis from 0 to $\frac{\pi}{2} + 0i$, C_3 from $\frac{\pi}{2} + 0i$ to $\frac{\pi}{2} + \frac{1}{2}\operatorname{Cosh}^{-1}\left(\frac{\alpha^2+1}{2\alpha}\right)i$, C_4 from $\frac{\pi}{2} + \frac{1}{2}\operatorname{Cosh}^{-1}\left(\frac{\alpha^2+1}{2\alpha}\right)i$ back to $\frac{\pi}{2} + 0i$, C_5 along the real axis from $\frac{\pi}{2}$ to π, and C_6 from $\pi + 0i$ to $\pi + \infty i$ (see Figure 8).

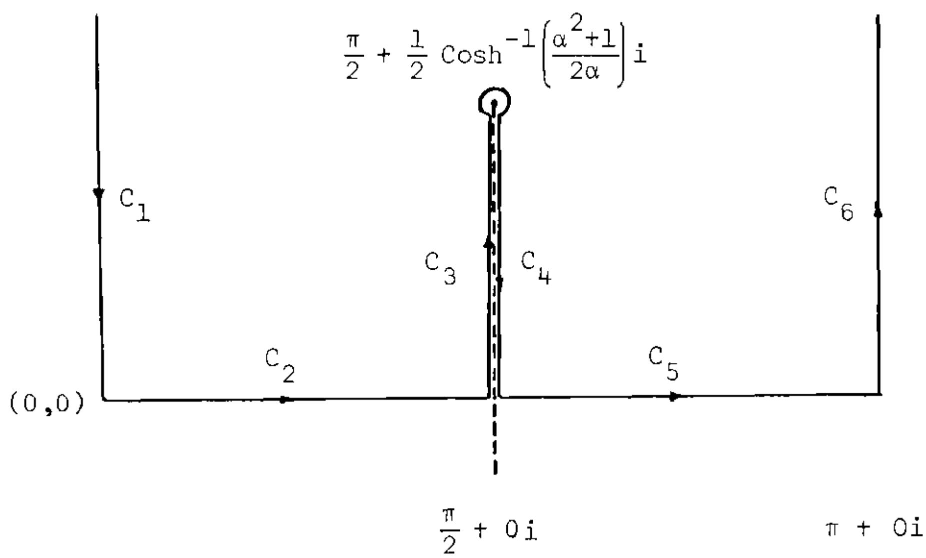


Figure 8. The Contour Γ

Let z be a complex variable restricted to the contour Γ and define the mapping

$$z = \frac{1}{2}\cos^{-1}\left[\frac{\alpha x^2}{2} - x(1+\alpha) + 1\right], \tag{77}$$

where the $\cos^{-1}$ is chosen so that $x=0$ maps onto $z=0$ and $x = 2\left(\frac{\alpha+1}{\alpha}\right)$ maps onto $z=\pi$. It is easily shown that this mapping maps the real line $-\infty<x<\infty$ onto the contour Γ. The mapping defined by

$$x = \frac{1}{\alpha}\left[(1+\alpha) - \sqrt{1+\alpha^2+2\alpha\cos 2z}\right] \tag{78}$$

on the complex plane cut from $\frac{\pi}{2} + \frac{1}{2}\operatorname{Cosh}^{-1}\left(\frac{\alpha^2+1}{2}\right)i$ to $\frac{\pi}{2} - \infty i$ (the principal square root chosen so that $z=0$ maps onto $x=0$) is the inverse mapping and maps Γ onto the real line. By use of (78) the recurrence (76) becomes

$$\begin{aligned}
K_0^{(\alpha,\beta)}(z) &= 1 \\
K_1^{(\alpha,\beta)}(z) &= -\frac{\beta}{2\alpha}\left[(1+\alpha) - \sqrt{1+\alpha^2+2\alpha\cos 2z}\right] + 1 \\
K_{2n}^{(\alpha,\beta)}(z) &= \left[(1-\alpha) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]K_{2n-1}^{(\alpha,\beta)}(z) - K_{2n-2}^{(\alpha,\beta)}(z), \quad n\geq 1, \\
K_{2n+1}^{(\alpha,\beta)}(z) &= \frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{\alpha}K_{2n}^{(\alpha,\beta)}(z) - K_{2n-1}^{(\alpha,\beta)}(z), \quad n\geq 1,
\end{aligned} \tag{79}$$

where $K_n^{(\alpha,\beta)}(z) = M_n^{(\alpha,\beta)}\left(x(z)\right)$, $n\geq 0$.

Lemma 11. Let $\{G_n^{(\alpha,\beta)}\}$ be the trigonometric polynomials determined by

$$G_0^{(\alpha,\beta)}(z) = 1$$

$$G_1^{(\alpha,\beta)}(z) = \beta\cos z + (1-\beta)\,\frac{\left[(1-\alpha) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{2\cos z}$$

$$G_{n+1}^{(\alpha,\beta)}(z) = 2(\cos z)G_n^{(\alpha,\beta)}(z) - G_{n-1}^{(\alpha,\beta)}(z), \quad n\geq 1.$$

Then the polynomials $K_n^{(\alpha,\beta)}(z)$ are given by

$$K_{2n}^{(\alpha,\beta)}(z) = G_{2n}^{(\alpha,\beta)}(z), \quad n\geq 0,$$

$$K_{2n+1}^{(\alpha,\beta)}(z) = \frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{2\alpha\cos z}\, G_{2n+1}^{(\alpha,\beta)}(z), \quad n\geq 0.$$

Proof. This result will be established by induction. For each integer $k\geq 1$ let T_k be the statement

"for $p=1,2,\ldots,k$ $\quad K_{2p-2}^{(\alpha,\beta)}(z) = G_{2p-2}^{(\alpha,\beta)}(z)$ and

$$K_{2p-1}^{(\alpha,\beta)}(z) = \frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{2\alpha\cos z}\, G_{2p-1}^{(\alpha,\beta)}(z)."$$

Clearly $K_0^{(\alpha,\beta)} = G_0^{(\alpha,\beta)}$, and

$$\frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{2\alpha\cos z}\, G_1^{(\alpha,\beta)}(z) = \frac{\beta}{2\alpha}\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]$$

$$+ (1-\beta)\,\frac{2\alpha(1+\cos 2z)}{4\alpha\cos^2 z}$$

$$= -\frac{\beta}{2\alpha}\left[(1+\alpha) - \sqrt{1+\alpha^2+2\alpha\cos 2z}\right] + 1 = K_1^{(\alpha,\beta)}(z);$$

hence T_1 is true. Suppose that T_k is true for some $k\geq 1$. Using recurrence (79) and the induction hypothesis, one finds that

$$K_{2n}^{(\alpha,\beta)}(z) = \frac{2\alpha(1+\cos 2z)}{2\alpha\cos z} G_{2n-1}^{(\alpha,\beta)}(z) - G_{2n-2}^{(\alpha,\beta)}(z)$$

$$= 2\cos z G_{2n-1}^{(\alpha,\beta)}(z) - G_{2n-2}^{(\alpha,\beta)}(z) = G_{2n}^{(\alpha,\beta)}(z)$$

and

$$K_{2n+1}^{(\alpha,\beta)}(z) = \frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{\alpha} G_{2n}^{(\alpha,\beta)}(z)$$

$$- \frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{2\alpha\cos z} G_{2n-1}^{(\alpha,\beta)}(z)$$

$$= \frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{2\alpha\cos z} \{2\cos z G_{2n}^{(\alpha,\beta)}(z) - G_{2n-1}^{(\alpha,\beta)}(z)\}$$

$$= \frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{2\alpha\cos z} G_{2n+1}^{(\alpha,\beta)}(z).$$

Thus T_k implies T_{k+1}, and the induction is completed.

Using a standard technique for solving difference equations, one easily finds that

$$G_n^{(\alpha,\beta)}(z) = \cos nz + (1-\beta)\,\frac{\sin nz}{\sin 2z}\;\{\sqrt{1+\alpha^2+2\alpha\cos 2z} - (\alpha+\cos 2z)\}, \quad n\geq 0.$$

Lemma 11 then yields the representation

$$K_{2n}^{(\alpha,\beta)}(x) = \cos 2nz + (1-\beta)\,\frac{\sin 2nz}{\sin 2z}\,\{\sqrt{1+\alpha^2+2\alpha\cos 2z} - (\alpha+\cos 2z)\}, \quad n\geq 0,$$

and

$$K_{2n+1}^{(\alpha,\beta)}(z) = \frac{\left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right]}{2\alpha\cos z}\left[\cos(2n+1)z\right. \tag{80}$$

$$\left. + (1-\beta)\,\frac{\sin(2n+1)z}{\sin 2z}\,\{\sqrt{1+\alpha^2+2\alpha\cos 2z} - (\alpha+\cos 2z)\}\right], \quad n\geq 0.$$

As previously indicated, the values $\alpha=1$ and $\beta=1$ or $\beta=2$ are the only choices for α and β for which the polynomials $M_n^{(\alpha,\beta)}$ are classical. For $\alpha=1$, Lemma 11 shows that

$$K_n^{(1,\beta)}(z) = G_n^{(1,\beta)}(z) = \cos nz + (1-\beta)\sin nz\,\frac{\sin(z/2)}{\cos(z/2)}, \quad n\geq 0.$$

Thus with $\alpha=1$, $\beta=1$ one finds that $K_n^{(1,1)}(z) = \cos nz$, $n\geq 0$; but from [23, p. 60], it follows that

$$M_n^{(1,1)}(x) = \frac{(-1)^n 4^n [n!]^2}{(2n)!}\,P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right), \quad n\geq 0,$$

where $P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$ is a Jacobi polynomial. Similarly, with $\alpha=1$, $\beta=2$ one finds that

$$K_n^{(1,2)}(z) = \frac{\cos\left(n + \frac{1}{2}\right)z}{\cos(z/2)}, \quad n\geq 0,$$

and it follows [23, p.60] that

$$M_n^{(1,2)}(z) = \frac{(-1)^n 4^n [n!]^2}{(2n)!} P_n^{\left(\frac{1}{2}, -\frac{1}{2}\right)}\left(\frac{x}{2} - 1\right), \quad n\geq 0.$$

Orthogonality Properties of $M_n^{(\alpha,\beta)}$

This section deals with the determination of the interval of orthogonality and the corresponding weight function (or integrator) for the polynomials $M_n^{(\alpha,\beta)}$. The case $\alpha=1$ is treated in detail to illustrate the procedure used. It will be shown that if $\beta\geq 1$ the polynomials $M_n^{(1,\beta)}(x)$ are orthogonal on $[0,4]$ with respect to a weight function, whereas if $\beta<1$ the polynomials are orthogonal on $\left[0, \frac{4}{\beta(2-\beta)}\right]$ with respect to a Stieltjes integrator which is constant on $\left[4, \frac{4}{\beta(2-\beta)}\right)$ and has a jump discontinuity at $x = \frac{4}{\beta(2-\beta)}$. Similar results are stated for the case $\beta=2$ and $\alpha>0$. For the case $\beta=1$, $\alpha>0$ there always exists a weight function.

Location of the Zeros of $M_n^{(1,\beta)}$

As a preliminary to determination of a weight function (or integrator), the location of the zeros of $M_n^{(1,\beta)}(x)$ is investigated. In this undertaking the corresponding trigonometric representation

$$K_n^{(1,\beta)}(z) = \cos nz + (1-\beta)\sin nz \frac{\sin(z/2)}{\cos(z/2)}$$

is used. It will be shown that if $\beta \geq 1$, then $K_n^{(1,\beta)}(z)$ has n zeros on $0<z<\pi$; that if $\beta<1$ and $n < \frac{1}{2(1-\beta)}$, then $K_n^{(1,\beta)}(z)$ has n zeros on $0<z<\pi$; and that if $\beta<1$ and $n > \frac{1}{2(1-\beta)}$, then $K_n^{(1,\beta)}(z)$ has $(n-1)$ zeros on $0<z<\pi$ and one zero on $C_6 = \{z:z=\pi + i\mu, \mu>0\}$. To this end, note that for $0<z<\pi$

$$K_n^{(1,\beta)}(z) = \sin nz\left[\cot nz + (1-\beta)\tan \frac{z}{2}\right],$$

and the zeros of $K_n^{(1,\beta)}(z)$ coincide with the zeros of

$$H_n(z) = \cot nz + (1-\beta)\tan \frac{z}{2}, \quad n\geq 1.$$

Let $z_k = \frac{k\pi}{n}$, $k=0,1,\ldots,n$, and consider $H_n(z)$ on (z_k,z_{k+1}), $k\leq(n-1)$. As $z \to z_k^+$, $H_n(z) \to +\infty$; furthermore $H_n\left(z_k + \frac{\pi}{2n}\right) = (1-\beta)\tan\left(\frac{z_k}{2} + \frac{\pi}{4n}\right)$, which has the sign of $(1-\beta)$. Thus if $\beta>1$ $H_n(z)$ changes sign between z_k and $z_k + \frac{\pi}{2n}$; so $H_n(z)$, and hence $K_n^{(1,\beta)}(z)$, has a zero on each subinterval (z_k,z_{k+1}), $k=0,1,\ldots,n-1$. This enumeration accounts for the n zeros of $K_n^{(1,\beta)}(z)$ on $(0,\pi)$. For $\beta<1$ it can be shown similarly that $K_n^{(1,\beta)}(z)$ has a zero on each subinterval (z_k,z_{k+1}), $k=0,1,\ldots,n-2$. A direct calculation shows that $K_n^{(1,\beta)}(z_{n-1})K_n^{(1,\beta)}(z_n) = 2n(1-\beta) - 1$. It follows for $\beta<1$ and $1 \leq n < \frac{1}{2(1-\beta)}$ that $K_n^{(1,\beta)}(z)$ also has a zero on (z_{n-1},z_n), thus accounting for the n zeros of $K_n^{(1,\beta)}(z)$ on $(0,\pi)$. For $\beta<1$ and $n > \frac{1}{2(1-\beta)}$, $K_n^{(1,\beta)}(z)$ has the same sign at z_{n-1} and z_n. Thus if

$n > \frac{1}{2(1-\beta)}$, then $K_n^{(1,\beta)}(z)$ either has no zero on (z_{n-1}, z_n) or has an even number of zeros on (z_{n-1}, z_n). Since $K_n^{(1,\beta)}(z)$ has only n zeros, and (n-1) of them are accounted for, there is no zero on (z_{n-1}, z_n) for $n > \frac{1}{2(1-\beta)}$.

It will now be shown that if $\beta<1$ and $n > \frac{1}{2(1-\beta)}$ the nth zero of $K_n^{(1,\beta)}(z)$ lies on $C_6 = \{z:z=\pi + i\mu,\ \mu>0\}$. For $z = \pi + i\mu$ the representation

$$K_n^{(1,\beta)}(z) = (-1)^n \cosh(n\mu)\coth\left(\frac{\mu}{2}\right)\left[\tanh \frac{\mu}{2} - (1-\beta)\tanh n\mu\right], \quad \mu>0$$

holds. Let $G_n(\mu) = \tanh \frac{\mu}{2} - (1-\beta)\tanh n\mu$. Since $G_n(0) = 0$, $G_n'(0) = \frac{1}{2} - (1-\beta)n < 0$, and $\lim_{\mu\to\infty} G_n(\mu) = \beta>0$, it follows that $G_n(\mu)$ has a zero for $\mu>0$ and consequently that $K_n^{(1,\beta)}(z)$ has its nth zero on C_6.

If for each n x_n is the largest zero of $M_n^{(1,\beta)}(x)$, then $\lim_{n\to\infty} x_n$ proves to be of considerable significance in the determination of the interval of orthogonality and, when $\beta<1$, the location of the point at which the jump in the integrator occurs. For any $\beta<1$ and $n > \frac{1}{2(1-\beta)}$, let μ_n denote the zero of $G_n(\mu)$. From the equation $G_n(\mu) = 0$ it follows that $\{\mu_n\}$ is an increasing sequence bounded above by $\hat{\mu}$ where $\tanh \frac{\hat{\mu}}{2} = (1-\beta)$. For $\hat{z} = \pi + i\hat{\mu}$, equation (78) (with $\alpha=1$) yields $\hat{x} = \frac{4}{\beta(2-\beta)}$. Thus if x_n, $n\geq 1$, denotes the largest zero of $M_n^{(1,\beta)}(x)$, then $\{x_n\}$ is an increasing sequence and

$$\bar{x} \triangleq \lim_{n\to\infty} x_n = \begin{cases} 4 & , \quad \beta\geq 1, \\ \dfrac{4}{\beta(2-\beta)} & , \quad \beta<1. \end{cases}$$

Determination of a Weight Function for $M_n^{(1,\beta)}$ for $\beta>1$

For $\beta>1$ it has been shown that all the zeros of $M_n^{(1,\beta)}(x)$ lie in the interval $(0,4)$ and that $(0,4)$ is the shortest interval of which this statement can be made. Since it is well known that all the zeros of each polynomial in any sequence of orthogonal polynomials lie in the interior of the interval of orthogonality, one is led to seek a weight function (if one exists) which is defined on $0\leq x\leq 4$.

A sequence of polynomials given by (1) is known to be a complete set of functions on a finite interval of orthogonality [23, p.40]. The supposition that $M_n^{(1,\beta)}(x)$ are orthogonal on $[0,4]$ with respect to a weight function might lead one to try to determine the unknown weight function as a Fourier series in the polynomials $M_n^{(1,\beta)}(x)$. For the polynomials $M_n^{(1,\beta)}(x)$, a more judicious choice of a Fourier series is possible. It may easily be shown that

$$M_n^{(1,\beta)}(x) = \frac{(-1)^n 4^n [n!]^2}{(2n)!}\left[\frac{\beta}{2}\, P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right) - \left(\frac{2-\beta}{2}\right)\frac{2n-1}{2}\, P_{n-1}^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right)\right]^{*}, \quad n\geq 1. \tag{81}$$

---

*The polynomials $P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}$ are the Jacobi polynomials described in [23, p.58ff.] with $\alpha = 1/2$, $\beta = -1/2$.

Furthermore the polynomials $P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right)$, $n\geq 0$, are orthogonal on $[0,4]$ with respect to the normalized weight function $\omega(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$, and the sequence $\left\{P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right)\right\}$ is a complete set of functions on $[0,4]$. Hence one conjectures that the polynomials $M_n^{(1,\beta)}(x)$, $n\geq 0$, are orthogonal on $[0,4]$ with respect to a normalized weight function which may be represented in the form

$$\rho(x) = \omega(x) \sum_{n=0}^{\infty} a_n q_n P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right), \tag{82}$$

where $q_n = \frac{(-1)^n 4^n [n!]^2}{(2n)!}$, $n\geq 0$, and a_n, $n\geq 0$, are generalized Fourier coefficients to be determined by imposing the orthogonality conditions which the $M_n^{(1,\beta)}(x)$ must satisfy; i.e.,

$$\int_0^4 \rho(x)dx = 1, \int_0^4 M_p^{(1,\beta)}(x)\rho(x)dx = 0, \quad p\geq 1. \tag{83}$$

Multiplication of (82) by $M_p^{(1,\beta)}(x)$, followed by integration over $[0,4]$ and use of (83), yields the infinite system of algebraic equations

$$\sum_{n=0}^{\infty} a_n \int_0^4 q_n P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right)\omega(x)dx = 1,$$

$$\sum_{n=0}^{\infty} a_n \int_0^4 q_n P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right)\left[\frac{\beta}{2} q_p P_p^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right) + \left(\frac{2-\beta}{2}\right) q_{p-1} P_{p-1}^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right)\right]\omega(x)dx = 0,$$

$p\geq 1$.

The integrals on the left may be evaluated by using the known orthogonality properties of $P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right)$. One finds that $\{a_n\}$ $(n\geq 0)$ must satisfy the equations

$$
\begin{aligned}
&a_0 = 1 \\
&\left(\frac{2-\beta}{2}\right)a_{p-1} + \left(\frac{\beta}{2}\right)a_p = 0, \quad p\geq 1.
\end{aligned}
\tag{84}
$$

The unique solution of (84) is

$$a_n = (-1)^n\left(\frac{2-\beta}{\beta}\right)^n, \quad n\geq 0, \tag{85}$$

and since $\left|\frac{2-\beta}{\beta}\right| < 1$ for all $\beta>1$, $\lim\limits_{n\to\infty} a_n = 0$ for $\beta>1$. Note that this result should have been expected from the Riemann-Lebesgue lemma, since the a_n $(n\geq 0)$ are Fourier coefficients. Note also that if $\beta<1$, then $|a_n| \to +\infty$ as $n\to\infty$.

Substitution of the coefficients (85) into (82) yields

$$\rho(x) = \omega(x)\sum_{n=0}^{\infty}(-1)^n\left(\frac{2-\beta}{\beta}\right)^n q_n P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{x}{2}-1\right).$$

With the substitution $1-\frac{x}{2} = \cos\theta$ and use of the identity $q_n P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(-\cos\theta) = \dfrac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\theta/2}$, it follows that

$$\rho(x) = \omega(x) \sum_{n=0}^{\infty} (-1)^n \left(\frac{2-\beta}{\beta}\right)^n \frac{\cos\left(n + \frac{1}{2}\right)\theta}{\cos\ \theta/2}$$

$$= \omega(x) \frac{2\beta}{4 - 4\beta + 2\beta^2 + 2\beta(2-\beta)\cos\theta}$$

$$= \frac{\beta}{\pi}\sqrt{\frac{4-x}{x}}\ \frac{1}{4 - \beta(2-\beta)x}\ ,\quad 0<x<4. \tag{86}$$

So if the polynomials $M_n^{(1,\beta)}(x)$ are orthogonal on [0,4] with respect to a weight function $\rho(x)$ of the form (82), then $\rho(x)$ is given by (86).

<u>Verification of the Orthogonality of $\{M_n^{(1,\beta)}\}$ for $\beta>1$</u>

The procedure used to determine the proposed weight function (86) is a formal one and hence the result requires direct verification. It should be observed that (for all $\beta>0$) ρ satisfies the non-negativity and integrability requirements to be imposed on any proposed weight function on [0,4].

<u>Lemma 12</u>. For $\beta>1$ the polynomials $M_n^{(1,\beta)}$ determined by (76) (with $\alpha=1$) are orthogonal on [0,4] with respect to the normalized weight function $\rho(x) = \frac{\beta}{\pi}\sqrt{\frac{4-x}{x}}\ \frac{1}{4 - \beta(2-\beta)x}$.

<u>*Proof*</u>. The proof of this result is by direct verification and makes use of the residue theorem. The details are tedious and only a summary of results is given. To evaluate the integrals $\int_0^4 M_n^{(1,\beta)}(x)M_m^{(1,\beta)}(x)\rho(x)dx$, make the change of variable $\cos\theta = 1 - \frac{x}{2}$ and then change the resulting trigonometric integrals to contour integrals around $|z| = 1$. Use of the residue theorem yields

$$\int_0^4 M_n^{(1,\beta)}(x)M_m^{(1,\beta)}(x)\rho(x)dx =$$

$$\frac{1}{4(\beta-2)} \operatorname*{Res}_{|z|<1} \left\{ \frac{(1+z)^2(z^{2n}+1)(z^{2m}+1) + (1-\beta)(z^2-1)(z^{2n+2m}-1)}{z^{n+m+1}\left[z - \frac{\beta-2}{\beta}\right]\left[z - \frac{\beta}{\beta-2}\right]} \right.$$

$$\left. + \frac{(1-\beta)^2(z-1)^2(z^{2n}-1)(z^{2m}-1)}{z^{n+m+1}\left[z - \frac{\beta-2}{\beta}\right]\left[z - \frac{\beta}{\beta-2}\right]} \right\}, \quad n\geq 0,\ m\geq 0.$$

The meromorphic function on the right has a pole of order $(n+m+1)$ at $z=0$ and simple poles at $z_1 = \frac{\beta - 2}{\beta}$ and $z_2 = \frac{\beta}{\beta - 2}$. For $\beta>1$ the simple pole $z_1 = \frac{\beta - 2}{\beta}$ lies inside $|z| = 1$. Evaluation of the residues at $z=0$ and $z = \frac{\beta - 2}{\beta}$ shows that

$$\int_0^4 M_n^{(1,\beta)}(x)M_m^{(1,\beta)}(x)\rho(x)dx = \begin{cases} 1, & n=m=0, \\ \frac{\beta}{2}, & n=m\neq 0, \\ 0, & n\neq m, \end{cases}$$

which completes the proof.

<u>Determination of an Integrator for $M_n^{(1,\beta)}$ for $\beta<1$</u>

Recall that for $\beta<1$ and $n > \frac{1}{2(1-\beta)}$ the largest zero x_n of $M_n^{(1,\beta)}(x)$ lies in the open interval $\left(4, \frac{4}{\beta(2-\beta)}\right)$ and $\lim_{n\to\infty} x_n = \frac{4}{\beta(2-\beta)}$. Hence $\left(0, \frac{4}{\beta(2-\beta)}\right)$ is the smallest open interval which, for every $n\geq 1$, contains all the zeros of $M_n^{(1,\beta)}(x)$. Consequently it is known that the interval of orthogonality contains the open interval $\left(0, \frac{4}{\beta(2-\beta)}\right)$. One might also note that $\left(4, \frac{4}{\beta(2-\beta)}\right)$ is the only open subinterval I of $\left(0, \frac{4}{\beta(2-\beta)}\right)$ of which it is true that for each $n\geq 1$ I contains at most one zero of $M_n^{(1,\beta)}(x)$. This property is known to be characteristic of any open subinterval of the interval of orthogonality on which the integrator is constant [23, p.50].

Even though $\rho(x) = \frac{\beta}{\pi}\sqrt{\frac{4-x}{x}}\,\frac{1}{4-\beta(2-\beta)x}$, $0<x<4$, cannot be a weight function for $\{M_n^{(1,\beta)}\}$ for $\beta<1$, one can evaluate $\int_0^4 M_n^{(1,\beta)}(x)M_m^{(1,\beta)}(x)\rho(x)dx$ by the residue theorem for $\beta<1$. The result of these calculations is

$$\int_0^4 M_n^{(1,\beta)}(x)M_m^{(1,\beta)}(x)\rho(x)dx = \begin{cases} 1 - \dfrac{2(1-\beta)}{(2-\beta)}, & n=m=0, \\[2mm] \dfrac{\beta}{2} - \dfrac{2(1-\beta)}{(2-\beta)}\left(\dfrac{\beta}{2-\beta}\right)^{2m}, & n=m\neq 0, \\[2mm] -(-1)^{n+m}\dfrac{2(1-\beta)}{(2-\beta)}\left(\dfrac{\beta}{2-\beta}\right)^{n+m}, & n\neq m. \end{cases} \tag{87}$$

As previously indicated the limit point $\bar{x}$ of the errant zero of $M_n^{(1,\beta)}(x)$ has a direct bearing on the location of the jump in the integrator.

<u>Lemma 13</u>. The value of the polynomial $M_n^{(1,\beta)}$ at $\bar{x} = \frac{4}{\beta(2-\beta)}$ is $M_n^{(1,\beta)}(\bar{x}) = (-1)^n\left(\frac{\beta}{2-\beta}\right)^n$, $n\geq 0$.

Proof. With $\alpha=1$ and $x=\bar{x}$ the recurrence (76) becomes

$$M_0^{(1,\beta)}(\bar{x}) = 1$$

$$M_1^{(1,\beta)}(\bar{x}) = -\frac{\beta}{2-\beta}$$

$$M_{n+1}^{(1,\beta)}(\bar{x}) = -\left(\frac{\beta}{2-\beta}+\frac{2-\beta}{\beta}\right)M_n^{(1,\beta)}(\bar{x}) - M_{n-1}^{(1,\beta)}(\bar{x}), \quad n\geq 1.$$

The assertion $M_n^{(1,\beta)}(\bar{x}) = (-1)^n\left(\frac{\beta}{2-\beta}\right)$, $n\geq 0$, may be established by a straightforward induction proof.

Combination of the result (87) with Lemma (13) yields

Lemma 14. For $0<\beta<1$ let ρ be defined on $\left(0, \frac{4}{\beta(2-\beta)}\right)$ by

$$\rho(x) = \begin{cases} \frac{\beta}{\pi}\sqrt{\frac{4-x}{x}}\,\frac{1}{4-\beta(2-\beta)x}, & 0<x<4, \\ 0 & , 4\leq x<\frac{4}{\beta(2-\beta)}, \end{cases}$$

and let α be the integrator defined on $\left[0, \frac{4}{\beta(2-\beta)}\right]$ by

$$\alpha(x) = \begin{cases} \int_0^x \rho(t)dt, & 0\leq x<\frac{4}{\beta(2-\beta)}, \\ \int_0^{\frac{4}{\beta(2-\beta)}} \rho(t)dt + \frac{2(1-\beta)}{(2-\beta)}, & x = \frac{4}{\beta(2-\beta)}. \end{cases}$$

Then the polynomials $\{M_n^{(1,\beta)}\}$ $(n\geq 0)$ are orthogonal on $\left[0, \frac{4}{\beta(2-\beta)}\right]$ with respect to the normalized integrator α.

Proof. Evaluation of the Stieltjes integral shows that

$$\int_0^{\frac{4}{\beta(2-\beta)}} M_n^{(1,\beta)}(x)M_m^{(1,\beta)}(x)d\alpha(x) =$$

$$\int_0^4 M_n^{(1,\beta)}(x)M_n^{(1,\beta)}(x)\rho(x)dx + \frac{2(1-\beta)}{(2-\beta)} M_n^{(1,\beta)}\left(\frac{4}{\beta(2-\beta)}\right)M_m^{(1,\beta)}\left(\frac{4}{\beta(2-\beta)}\right),$$

$$n\geq 0,\ m\geq 0.$$

Use of (87) and Lemma 13 yields

$$\int_0^{\frac{4}{\beta(2-\beta)}} M_n^{(1,\beta)}(x)M_m^{(1,\beta)}(x)d\alpha(x) = \begin{cases} 1, & n=m=0 \\ \frac{\beta}{2}, & n=m\neq 0 \\ 0, & n\neq m, \end{cases}$$

which completes the proof.

Orthogonality of the Polynomials $M_n^{(\alpha,2)}$

A procedure not dissimilar to that used in the preceding analysis of $M_n^{(1,\beta)}$ may be used to determine the interval of orthogonality and weight function (or integrator) for the polynomials $M_n^{(\alpha,2)}$ (i.e., $\alpha>0$, $\beta=2$ in recurrence (76)). It will be shown that for $0<\alpha<1$ the polynomials $M_n^{(\alpha,2)}$ ($n\geq 0$) are orthogonal on $\left[0,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ with respect to a weight function which is identically zero on $[2,\ 2/\alpha]$. For $\alpha>1$ the polynomials $M_n^{(\alpha,2)}$ are orthogonal on $\left[0,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ with respect to a Stieltjes integrator with a jump located at $\bar{x} = \left(\frac{\alpha+1}{\alpha}\right)$. The major aspects of this analysis are summarized here.

The location of the zeros of $M_n^{(\alpha,2)}(x)$ may be determined by using the corresponding trigonometric representation $K_n^{(\alpha,2)}(z)$ obtained from (80) with $\beta=2$. One finds that

$$K_{2n}^{(\alpha,2)}(z) = \frac{\sin(2n+1)z}{\sin z} + \left[(\alpha-1) - \sqrt{1+\alpha^2+2\alpha\cos 2z}\right] \frac{\sin 2nz}{\sin 2z}, \quad n\geq 0,$$

$$K_{2n+1}^{(\alpha,2)}(z) = -\frac{\sin(2n+1)z}{\sin z} + \left[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}\right] \frac{\sin(2n+2)z}{\alpha \sin 2z}, \quad n\geq 0.$$

It can be shown that for $0<\alpha<1$ $M_n^{(\alpha,2)}(x)$ has $\left[\frac{n}{2}\right]$ zeros in the open interval $(0,2)$ and $\left[\frac{n+1}{2}\right]$ zeros in the open interval $\left(\frac{2}{\alpha}, 2\left(\frac{\alpha+1}{\alpha}\right)\right)$; that if $\alpha>1$ and $n < \frac{1}{(\alpha-1)}$, then $M_n^{(\alpha,2)}(x)$ has $\left[\frac{n}{2}\right]$ zeros in the open interval $\left(0, \frac{2}{\alpha}\right)$ and $\left[\frac{n+1}{2}\right]$ zeros in the open interval $\left(2,2\left(\frac{\alpha+1}{\alpha}\right)\right)$; and that if $\alpha>1$ and $n > \frac{1}{(\alpha-1)}$, then $M_n^{(\alpha,2)}(x)$ has $\left[\frac{n}{2}\right]$ zeros in the open interval $\left(0, \frac{2}{\alpha}\right)$, $\left[\frac{n-1}{2}\right]$ zeros in the open interval $\left(2,2\left(\frac{\alpha+1}{\alpha}\right)\right)$ and one zero in the open interval $\left(\frac{\alpha+1}{\alpha},2\right)$. Furthermore if x_{n+1} denotes the $(n+1)$th zero of $M_{2n}^{(\alpha,2)}(x)$ $\left(\text{or of } M_{2n+1}^{(\alpha,2)}(x)\right)$, then $\{x_{n+1}\}$ is a decreasing sequence bounded below and

$$\lim_{n\to\infty} x_{n+1} = \begin{cases} \frac{2}{\alpha}, & 0<\alpha<1, \\ \left(\frac{1+\alpha}{\alpha}\right), & \alpha>1. \end{cases}$$

The location of the zeros of $M_n^{(\alpha,2)}(x)$ and the fact that for $\alpha<1$ the open interval $\left(2, \frac{2}{\alpha}\right)$ is the only subinterval of $\left(0, 2\left(\frac{\alpha+1}{\alpha}\right)\right)$ which contains no zero of $M_n^{(\alpha,2)}(x)$ suggest that one should seek a weight function (if one exists) which may be represented in the form

$$\omega(x) = \sum_{n=0}^{\infty} a_n M_n^{(\alpha,2)}(x)$$

on $(0,2] \cup \left[\frac{2}{\alpha}, 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ and which is identically zero on $\left[2, \frac{2}{\alpha}\right]$. With this supposition it is found that imposing the orthogonality conditions

$$\int_0^{2\left(\frac{\alpha+1}{\alpha}\right)} \omega(x)dx = 1, \int_0^{2\left(\frac{\alpha+1}{\alpha}\right)} M_p^{(\alpha,2)}(x)\omega(x)dx = 0, \quad p\geq 1,$$

leads to an infinite system of algebraic equations for the generalized Fourier coefficients a_n $(n\geq 0)$. The resulting system of algebraic equations can be shown to have a unique bounded solution [13, pp. 20ff.] which may be obtained by using the method of successive truncations. The use of these coefficients results in the proposed normalized weight function

$$\omega(x) = \begin{cases} \dfrac{1}{2\pi} \dfrac{\sqrt{\left(\frac{2}{\alpha} - x\right)(2-x)\left[2\left(\frac{\alpha+1}{\alpha}\right) - x\right]}}{\sqrt{x}\left|\frac{\alpha+1}{\alpha} - x\right|}, & 0<x<2,\ \frac{2}{\alpha}<x<2\left(\frac{\alpha+1}{\alpha}\right) \\ 0, & 2\leq x\leq \frac{2}{\alpha}\ . \end{cases} \tag{88}$$

A direct but lengthy evaluation using contour integrals proves that for $0<\alpha<1$

$$\int_0^{2\left(\frac{\alpha+1}{\alpha}\right)} M_n^{(\alpha,2)}(x)M_m^{(\alpha,2)}(x)\omega(x)dx = \begin{cases} 1, & n=m=2k,\ k\geq 0 \\ \frac{1}{\alpha}, & n=m=2k+1,\ k\geq 0 \\ 0, & n\neq m, \end{cases}$$

and hence $\{M_n^{(\alpha,2)}\}$ are orthogonal on $\left[0,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ with respect to ω.

For $\alpha>1$ the residue theorem yields the results

$$\int_0^{2\left(\frac{\alpha+1}{\alpha}\right)} M_{2n}^{(\alpha,2)}(x)M_{2m+1}^{(\alpha,2)}(x)\omega(x)dx = -\left(\frac{\alpha-1}{\alpha}\right)\left(-\frac{1}{\alpha}\right)^{n+m+1},\quad n\geq 0,\ m\geq 0,$$

$$\int_0^{2\left(\frac{\alpha+1}{\alpha}\right)} M_{2n}^{(\alpha,2)}(x)M_{2m}^{(\alpha,2)}(x)\omega(x)dx = \delta_{mn} - \left(\frac{\alpha-1}{\alpha}\right)\left(-\frac{1}{\alpha}\right)^{m+n},\quad n\geq 0,\ m\geq 0,$$

$$\int_0^{2\left(\frac{\alpha+1}{\alpha}\right)} M_{2n+1}^{(\alpha,2)}(x)M_{2m+1}^{(\alpha,2)}(x)\omega(x)dx = \frac{\delta_{mn}}{\alpha} - \left(\frac{\alpha-1}{\alpha}\right)\left(-\frac{1}{\alpha}\right)^{m+n+2},\quad n\geq 0,\ m\geq 0.$$

By a straightforward calculation using recurrence (76) (with $\beta=2$) it is found that

$$M_{2n}^{(\alpha,2)}\left(\frac{\alpha+1}{\alpha}\right) = \left(-\frac{1}{\alpha}\right)^n,\quad n\geq 0,$$

$$M_{2n+1}^{(\alpha,2)}\left(\frac{\alpha+1}{\alpha}\right) = \left(-\frac{1}{\alpha}\right)^{n+1},\quad n\geq 0.$$

The preceding remarks may be summarized to yield

Lemma 15. For $\alpha>1$ let ω be defined on $\left(0,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right)$ by

$$\omega(x) = \begin{cases} \dfrac{1}{2\pi}\dfrac{\sqrt{\left(\frac{2}{\alpha}-x\right)(2-x)\left[2\left(\frac{\alpha+1}{\alpha}\right)-x\right]}}{\sqrt{x}\left|\frac{\alpha+1}{\alpha}-x\right|}, & 0<x<\frac{2}{\alpha},\ 2<x<2\left(\frac{\alpha+1}{\alpha}\right), \\ 0 & ,\ \frac{2}{\alpha}\le x\le 2, \end{cases}$$

and let σ be defined on $\left[0,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ by

$$\sigma(x) = \begin{cases} \int_0^x \omega(t)dt, & 0 < x < \left(\frac{\alpha+1}{\alpha}\right) \\ \int_0^x \omega(t)dt + \left(\frac{\alpha-1}{\alpha}\right), & \left(\frac{\alpha+1}{\alpha}\right) \le x \le 2\left(\frac{\alpha+1}{\alpha}\right). \end{cases}$$

Then $M_n^{(\alpha,2)}$ are orthogonal on $\left[0,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ with respect to the normalized integrator σ.

Orthogonality of the Polynomials $M_n^{(\alpha,1)}$

The trigonometric representation (80) (with $\beta=1$) and the change of variable (77) may be used to obtain the representation [23, p.60]

$$M_{2n}^{(\alpha,1)}(x) = \frac{4^n(n!)^2}{(2n)!}\,P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{\alpha x^2}{2}-(\alpha+1)x+1\right),\quad n\ge 0, \qquad (89)$$

$$M_{2n+1}^{(\alpha,1)}(x) = \frac{4^n(n!)^2}{(2n)!}\left(1-\frac{x}{2}\right)P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}\left(\frac{\alpha x^2}{2}-(\alpha+1)x+1\right), \quad n\geq 0.$$

With the representation (89) one can prove

Lemma 16. The polynomials $\{M_n^{(\alpha,1)}\}$ $(n\geq 0)$ are orthogonal on the interval $\left[0,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ with respect to the normalized weight function

$$w(x) = \begin{cases} \dfrac{1}{\pi}\dfrac{\left(\frac{2}{\alpha}-x\right)}{\sqrt{x\left(2\left(\frac{\alpha+1}{\alpha}\right)-x\right)(2-x)\left(\frac{2}{\alpha}-x\right)}}, & 0<x<\left(\frac{\alpha+1}{\alpha}\right)\left(1-\left|\frac{\alpha-1}{\alpha+1}\right|\right), \\[2ex] 0, & \left(\frac{\alpha+1}{\alpha}\right)\left(1-\left|\frac{\alpha-1}{\alpha+1}\right|\right)\leq x\leq\left(\frac{\alpha+1}{\alpha}\right)\left(1+\left|\frac{\alpha-1}{\alpha+1}\right|\right), \\[2ex] \dfrac{1}{\pi}\dfrac{\left(x-\frac{2}{\alpha}\right)}{\sqrt{x\left(2\left(\frac{\alpha+1}{\alpha}\right)-x\right)(x-2)\left(x-\frac{2}{\alpha}\right)}}, & \left(\frac{\alpha+1}{\alpha}\right)\left(1+\left|\frac{\alpha-1}{\alpha+1}\right|\right)<x<2\left(\frac{\alpha+1}{\alpha}\right). \end{cases}$$

Proof. Because of the different representations of $M_{2n}^{(\alpha,1)}$ and $M_{2n+1}^{(\alpha,1)}$, the proof falls naturally into three cases to show

$$\int_0^{2\left(\frac{\alpha+1}{\alpha}\right)} M_p^{(\alpha,1)}(x)M_k^{(\alpha,1)}(x)w(x)dx = \begin{cases} 1, & p=k=0, \\ 0, & p\neq k, \\ \frac{1}{2\alpha}, & p=k \text{ (both odd)}, \\ \frac{1}{2}, & p=k \text{ (both even)}. \end{cases}$$

The proof for p and k both even will be given in detail, the other two cases being similar. For notational convenience let $q_n = \frac{4^n(n!)^2}{(2n)!}$; make the change of variable $s = \frac{\alpha x - (\alpha+1)}{(\alpha+1)}$; and let $\gamma = \left(\frac{\alpha-1}{\alpha+1}\right)$. Then

$$\int_0^{2\left(\frac{\alpha+1}{\alpha}\right)} M_{2n}^{(\alpha,1)}(x) M_{2m}^{(\alpha,1)}(x) w(x)dx =$$

$$q_n q_m \int_{-1}^{-|\gamma|} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\{r(s)\} P_m^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\{r(s)\} \frac{[-(\gamma+s)]}{\sqrt{(1-s^2)(s^2-\gamma^2)}} ds$$

$$+ q_n q_m \int_{|\gamma|}^{1} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\{r(s)\} P_m^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\{r(s)\} \frac{[\gamma+s]}{\sqrt{(1-s^2)(s^2-\gamma^2)}} ds$$

$$= q_n q_m \int_{|\gamma|}^{1} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\{r(s)\} P_m^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\{r(s)\} \frac{(2s)ds}{\sqrt{(1-s^2)(s^2-\gamma^2)}} ,$$

where $r(s) = \frac{(1+\alpha)^2}{2\alpha} s^2 - \frac{1+\alpha^2}{2\alpha}$. In this last integral make the substitution $t = r(s) = \frac{(1+\alpha)^2}{2\alpha} s^2 - \frac{1+\alpha^2}{2\alpha}$ to obtain

$$q_n q_m \int_{|\gamma|}^{1} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\{r(s)\} P_m^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(r,s) \frac{2sds}{\sqrt{(1-s^2)(s^2-\gamma^2)}}$$

$$= q_n q_m \int_{-1}^{1} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(t) P_m^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(t)(1+t)^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt,$$

$$= \begin{cases} 0, & n\neq m, \\ 1, & n=m=0, \\ \frac{1}{2}, & n=m\neq 0. \end{cases}$$

The other two cases may be shown similarly, which completes the proof.

The Non-Classical Polynomials $A_n^{(\alpha)}$

For any $\alpha>0$ let $A_n^{(\alpha)}$ $(n\geq 1)$ be the polynomials generated by the recurrence relation*

$$\begin{aligned}
A_1^{(\alpha)}(x) &= 1 \\
A_2^{(\alpha)}(x) &= (-\alpha x+2) \\
A_{2n+1}^{(\alpha)}(x) &= (-x+2)A_{2n}^{(\alpha)}(x) - A_{2n-1}^{(\alpha)}(x), \quad n\geq 1 \\
A_{2n+2}^{(\alpha)}(x) &= (-\alpha x+2)A_{2n+1}^{(\alpha)}(x) - A_{2n}^{(\alpha)}(x), \quad n\geq 1.
\end{aligned} \tag{90}$$

The only value of α for which the polynomials $A_n^{(\alpha)}$ are classical polynomials is $\alpha=1$. This result may be established in much the same way as for the polynomials $M_n^{(\alpha,\beta)}$. One finds that

$$g_1(2n+1) = \frac{8n(\alpha-1)(2\alpha^2-3\alpha+2)}{3\alpha^2} + \frac{8(\alpha^2+1)(\alpha-1)}{3\alpha^2}, \quad n\geq 0.$$

Thus the condition that $g_1(n) = 0$, $n\geq 0$, implies that $\alpha=1$ is a necessary condition for $A_n^{(\alpha)}$ $(n\geq 1)$ to be classical polynomials. That $\alpha=1$ is also a sufficient condition follows from [11].

Use of the transformation (78) changes the recurrence (90) to the corresponding trigonometric recurrence

---

*Note that the polynomial $A_n^{(\alpha)}$ $(n\geq 1)$ has degree $(n-1)$.

$$
\begin{aligned}
H_1^{(\alpha)}(z) &= 1 \\
H_2^{(\alpha)}(z) &= [(1-\alpha) + \sqrt{1+\alpha^2+2\alpha\cos 2z}] \\
H_{2n+1}^{(\alpha)}(z) &= \frac{[(\alpha-1) + \sqrt{1+\alpha^2+2\alpha\cos 2z}]}{\alpha} H_{2n}^{(\alpha)}(z) - H_{2n-1}^{(\alpha)}(z), \quad n\geq 1 \\
H_{2n+2}^{(\alpha)}(z) &= [(1-\alpha) + \sqrt{1+\alpha^2+2\alpha\cos 2z}]H_{2n+1}^{(\alpha)}(z) - H_{2n}^{(\alpha)}(z), \quad n\geq 1,
\end{aligned}
\tag{91}
$$

where $H_n^{(\alpha)}(z) = A_n^{(\alpha)}\left(x(z)\right)$. A straightforward induction proof in which (91) is used establishes the representation

$$
H_{2n}^{(\alpha)}(z) = [(1-\alpha) + \sqrt{1+\alpha^2+2\alpha\cos 2z}]\,\frac{\sin 2nz}{\sin 2z}, \quad n\geq 1,
$$

$$
H_{2n+1}^{(\alpha)}(z) = \frac{\sin(2n+1)z}{\sin z}, \quad n\geq 0.
$$

This trigonometric representation and the change of variable (77) may be used to obtain the representation [23, p.60]

$$
\begin{aligned}
A_{2n+1}^{(\alpha)}(x) &= \frac{4^n(n!)^2}{(2n)!}\, P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{\alpha x^2}{2} - x(1+\alpha) + 1\right), \quad n\geq 0 \\
A_{2n+2}^{(\alpha)}(x) &= \frac{2\cdot 4^n[(n+1)!]^2}{(2n+2)!}\,(2-\alpha x)P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{\alpha x^2}{2} - x(1+\alpha) + 1\right), \quad n\geq 0.
\end{aligned}
\tag{92}
$$

Use of the representation (92) and a proof analogous to the proof of Lemma 16 establish

Lemma 17. The polynomials $\{A_n^{(\alpha)}\}$ $(n\geq 1)$ are orthogonal on $\left[0,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right]$ with respect to the normalized weight function

$$r(x) = \begin{cases} \dfrac{\alpha}{2\pi}(2-x)\sqrt{\dfrac{x\left[2\left(\frac{\alpha+1}{\alpha}\right)-x\right]}{(2-x)\left(\frac{2}{\alpha}-x\right)}}, & 0\leq x<\left(\dfrac{1+\alpha}{\alpha}\right)\left(1-\left|\dfrac{\alpha-1}{\alpha+1}\right|\right) \\ 0, & \left(\dfrac{1+\alpha}{\alpha}\right)\left(1-\left|\dfrac{\alpha-1}{\alpha+1}\right|\right)\leq x\leq\left(\dfrac{1+\alpha}{\alpha}\right)\left(1+\left|\dfrac{\alpha-1}{\alpha+1}\right|\right) \\ \dfrac{\alpha}{2\pi}(x-2)\sqrt{\dfrac{x\left[2\left(\frac{\alpha+1}{\alpha}\right)-x\right]}{(x-2)\left(x-\frac{2}{\alpha}\right)}}, & \left(\dfrac{1+\alpha}{\alpha}\right)\left(1+\left|\dfrac{\alpha-1}{\alpha+1}\right|\right)<x\leq 2\left(\dfrac{\alpha+1}{\alpha}\right). \end{cases}$$

The location of the zeros of $A_n^{(\alpha)}$ can be determined from (92) as follows. The Jacobi polynomial $P_n^{(1/2,-1/2)}(t)$ is known to have exactly n zeros on the open interval $-1<t<1$. It follows that $A_{2n+1}^{(\alpha)}(x)$ has exactly n zeros for $-1 < \frac{\alpha x^2}{2} - x(1+\alpha) + 1 < 1$ or equivalently for $0 < \frac{\alpha x^2}{2} - x(1+\alpha) + 2 < 2$. From this inequality it follows that $A_{2n+1}^{(\alpha)}(x)$ has n zeros on the open interval $(0,A)$ and n zeros on the open interval $\left(B,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right)$ where $A = \min\{2, \frac{2}{\alpha}\}$ and $B = \max\{2, \frac{2}{\alpha}\}$. Similarly $A_{2n+2}^{(\alpha)}(x)$ has n zeros on $(0,A)$, n zeros on $\left(B,\ 2\left(\frac{\alpha+1}{\alpha}\right)\right)$, and a zero at $x = \frac{2}{\alpha}$.

CHAPTER V

APPLICATIONS OF GENERALIZED WEIGHT FUNCTIONS TO THE SOLUTION OF SOME ISOTOPIC IMPURITY PROBLEMS

The analysis of linear chains with various types of impurities has been the subject of considerable investigation in recent years [6,8,9,17,19,20,24,25]. The general problem of random impurities randomly distributed throughout the chain remains unsolved yet promises to be a rewarding area for future investigations. The purpose of this chapter is to present solutions of the equations of motion for three examples of infinite linear chains with isotopic impurities. The first of these examples (the infinite diatomic chain) is analyzed in detail and illustrates the techniques developed in Chapter II. The remaining two examples are accompanied by a complete summary of the components required to construct the solution.

An Infinite Diatomic Chain

Consider the physical system consisting of masses m and αm alternately arranged to form an infinite chain in which each mass is coupled to each of its nearest neighbors by a linear restoring force with proportionality constant k (see Figure 9a). Without loss of generality one may suppose that $\alpha>1$ and may designate a mass m as the middle of the infinite chain. Let x_n $(-\infty<n<\infty)$ denote the displacement (measured from equilibrium) of the mass located $|n|$ positions from the middle of the chain (to the right if $n>0$ and to the left if $n<0$), and suppose that the mass in the kth position ($k\geq0$) is given an initial displacement a and an initial velocity b and that all other masses are initially stationary in their respective equilibrium positions.

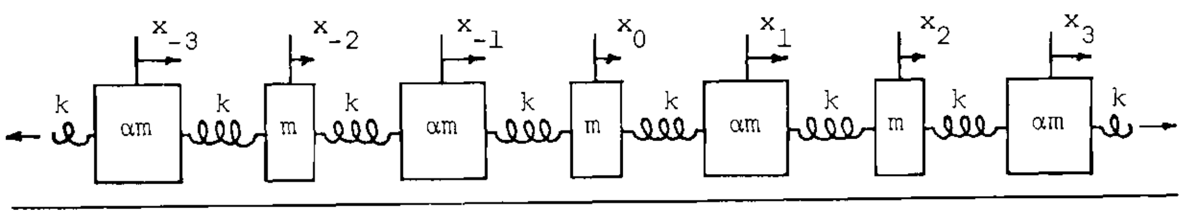


Figure 9a. An Infinite Diatomic Chain

The equations of motion of the system are

$$
\begin{aligned}
m\ddot{x}_{2n} &= k(x_{2n-1}-x_{2n}) + k(x_{2n+1}-x_{2n}), \quad n\leq-1 \\
\alpha m\ddot{x}_{2n-1} &= k(x_{2n-2}-x_{2n-1}) + k(x_{2n}-x_{2n-1}), \quad n\leq0 \\
m\ddot{x}_{0} &= k(x_{-1}-x_{0}) + k(x_{1}-x_{0}) \\
\alpha m\ddot{x}_{2n+1} &= k(x_{2n}-x_{2n+1}) + k(x_{2n+2}-x_{2n+1}), \quad n\geq0 \\
m\ddot{x}_{2n} &= k(x_{2n-1}-x_{2n}) + k(x_{2n+1}-x_{2n}), \quad n\geq1
\end{aligned}
\tag{93}
$$

subject to the initial conditions

$$
\begin{aligned}
x_k(0) &= a, \quad \dot{x}_k(0) = b, \\
x_n(0) &= 0, \quad \dot{x}_n(0) = 0, \quad n\neq k.
\end{aligned}
\tag{93.1}
$$

When the system (93) is written in the notation of (12) and the coefficients are identified, one finds that

$$A_{2n} = -\frac{m}{k}\ (n\geq 0),\ \ A_{2n+1} = -\frac{\alpha m}{k}\ \ (n\geq 0),$$

$$B_n = 2,\ \ n\geq 0,$$

$$C_n = 1,\ \ n\geq 1.$$

Thus from equation (18) the symmetric part $\{w_n\}$ of the solution satisfies the system

$$-\left(\frac{m}{2k}\ddot{w}_0+w_0\right) + w_1 = 0$$

$$w_{2n} - \left(\frac{\alpha m}{k}\ddot{w}_{2n+1}+2w_{2n+1}\right) + w_{2n+2} = 0,\ \ n\geq 0, \tag{94}$$

$$w_{2n-1} - \left(\frac{m}{k}\ddot{w}_{2n}+2w_{2n}\right) + w_{2n+1} = 0,\ \ n\geq 1,$$

subject to the initial conditions

$$w_k(0) = (1+\delta_{0k})\ \frac{a}{2},\ \ \dot{w}_k(0) = (1+\delta_{0k})\ \frac{b}{2},$$

$$w_n(0) = 0,\ \ \dot{w}_n(0) = 0,\ \ \ n\neq k. \tag{94.1}$$

From equation (19) the antisymmetric part $\{z_n\}$ of the solution satisfies the system

$$z_0 \equiv 0$$

$$-\left(\frac{\alpha m}{k}\,\ddot{z}_1+2z_1\right)+z_2=0$$

$$z_{2n-1}-\left(\frac{m}{k}\,\ddot{z}_{2n}+2z_{2n}\right)+z_{2n+1}=0,\quad n\geq 1, \tag{95}$$

$$z_{2n}-\left(\frac{\alpha m}{k}\,\ddot{z}_{2n+1}+2z_{2n+1}\right)+z_{2n+2}=0,\quad n\geq 1,$$

subject to the initial conditions

$$z_k(0)=(1-\delta_{0k})\,\frac{a}{2},\quad \dot{z}_k(0)=(1-\delta_{0k})\,\frac{b}{2}\,,$$

$$z_n(0)=0,\ \dot{z}_n(0)=0,\quad n\neq k.$$

Since both differential systems (94) and (95) are indexed for $n\geq 0$, each of these systems may be interpreted as mathematical models for half-infinite systems. A physical system corresponding to (94) may be visualized as obtained from the infinite system (Figure 9a) by using one-half the middle mass (unconstrained on the left) and the remainder of the right side of the infinite system. This half-infinite physical system will be referred to as the symmetric part of the infinite system (see Figure 9b). A physical system corresponding to (95) may be realized by holding the mass in the middle position fixed and considering that part of the infinite chain to the right of the middle mass.

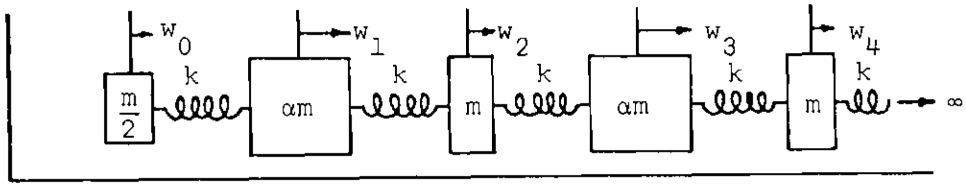


Figure 9b. The Symmetric Part of the Infinite Diatomic Chain

This half-infinite physical system will be referred to as the antisymmetric part of the infinite system (see Figure 9c).

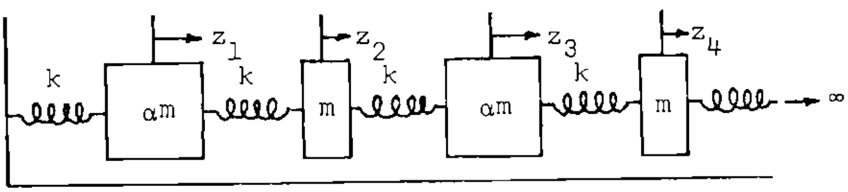


Figure 9c. The Antisymmetric Part of the Infinite Diatomic Chain

Theorem 4 shows that the solution of the infinite system (93), (93.1) is obtained by superposing the symmetric and antisymmetric solutions from (94), (94.1) and (95), (95.1), respectively.

The solutions of (94), (94.1) and (95), (95.1) can be deduced by identifying the appropriate polynomials and corresponding weight functions. The polynomials associated with the symmetric mode $\{w_n\}$ are given by

$$R_0(x) = 1 \tag{96}$$

$$R_1(x) = \left(-\frac{m}{2k}\,x+1\right)$$

$$R_{2n}(x) = \left(-\frac{\alpha m}{k}x+2\right)R_{2n-1}(x) - R_{2n-2}(x), \quad n\geq 1,$$

$$R_{2n+1}(x) = \left(-\frac{m}{k}x+2\right)R_{2n}(x) - R_{2n-1}(x), \quad n\geq 1.$$

Comparison of recurrence (96) with (76) shows that $R_n(x) = M_n^{(\alpha,1)}\left(\frac{m}{k}x\right)$, and from (89) it follows that

$$R_{2n}(x) = \frac{4^n(n!)^2}{(2n)!}P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{\alpha m^2x^2}{2k^2} - \frac{mx}{k}(\alpha+1) + 1\right), \quad n\geq 0,$$

$$R_{2n+1}(x) = \frac{4^n(n!)^2}{(2n)!}\left(1 - \frac{mx}{2k}\right)P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}\left(\frac{\alpha m^2x^2}{2k^2} - \frac{mx}{k}(\alpha+1) + 1\right), \quad n\geq 0.$$

The corresponding weight function is given in Lemma 16 as

$$\rho(x) = \frac{d\alpha(x)}{dx} = \begin{cases} \dfrac{1}{\pi}\dfrac{\left|x - \frac{2k}{\alpha m}\right|}{\sqrt{x\left(\frac{2k}{m}\left(\frac{1+\alpha}{\alpha}\right)-x\right)\left(x - \frac{2k}{\alpha m}\right)\left(x - \frac{2k}{m}\right)}}, & 0<x<\frac{2k}{\alpha m},\ \frac{2k}{m}<x<\frac{2k}{m}\left(\frac{1+\alpha}{\alpha}\right) \\ 0, & \frac{2k}{\alpha m}\leq x\leq\frac{2k}{m}, \end{cases}$$

and the normalizing factors, as determined by Lemma 1, are found to be

$$\gamma_0 = 1, \ \gamma_{2n} = \frac{1}{2}, \quad n\geq 1,$$

$$\gamma_{2n+1} = \frac{1}{2\alpha}, \quad n\geq 0.$$

For the antisymmetric mode $\{z_n\}$ the appropriate polynomials are given by

$$
\begin{aligned}
Q_0 &= 0, \quad Q_1(x) = 1 \\
Q_2(x) &= \left(-\frac{\alpha m}{k}x+2\right) \\
Q_{2n+1}(x) &= \left(-\frac{m}{k}x+2\right)Q_{2n}(x) - Q_{2n-1}(x), \quad n\geq 1, \\
Q_{2n+2}(x) &= \left(-\frac{\alpha m}{k}x+2\right)Q_{2n+1}(x) - Q_{2n}(x), \quad n\geq 1.
\end{aligned}
\tag{97}
$$

Comparison of (97) with (90) shows that $Q_n(x) = A_n^{(\alpha)}\left(\frac{m}{k}x\right)$ and from (92) it follows that

$$Q_{2n+1}(x) = \frac{4^n(n!)^2}{(2n)!}P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{\alpha m^2x^2}{2k^2} - \frac{mx}{k}(\alpha+1)+1\right), \quad n\geq 0,$$

$$Q_{2n+2}(x) = \frac{4^{n+1}[(n+1)!]^2}{(2n+2)!}\left(1-\frac{\alpha m}{2k}x\right)P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{\alpha m^2x^2}{2k^2} - \frac{mx}{k}(1+\alpha)+1\right), \quad n\geq 1.$$

Lemma 17 gives the corresponding weight function

$$\rho(x) = \frac{d\omega(x)}{dx} = \begin{cases} \dfrac{m^2\alpha}{2\pi k^2}\left|x-\dfrac{2k}{m}\right|\sqrt{\dfrac{x\left(\frac{2k}{m}\left(\frac{1+\alpha}{\alpha}\right)-x\right)}{\left(x-\frac{2k}{m}\right)\left(x-\frac{2k}{\alpha m}\right)}}, & 0<x<\dfrac{2k}{\alpha m},\ \dfrac{2k}{m}<x<\dfrac{2k}{m}\left(\dfrac{1+\alpha}{\alpha}\right), \\ 0 & , \dfrac{2k}{\alpha m}\leq x\leq\dfrac{2k}{m}, \end{cases}$$

and Lemma 1 yields the normalizing factors

$$\xi_0 = 1, \quad \xi_{2n} = \alpha, \quad n\geq 1,$$

$$\xi_{2n+1} = 1, \quad n\geq 0.$$

Theorem 5 yields the solution of the infinite system (93), (93.1):

$$x_n(t) = \frac{(1+\delta_{0k})}{2}\int_0^{2\frac{k}{m}\left(\frac{\alpha+1}{\alpha}\right)} R_n(x)\,\frac{R_k(x)}{\gamma_k}\left[a\cos\sqrt{x}t + b\frac{\sin\sqrt{x}t}{\sqrt{x}}\right]d\alpha(x)$$

$$+\frac{\operatorname{sgn}(kn)(1-\delta_{0k})}{2}\int_0^{\frac{2k}{m}\left(\frac{\alpha+1}{\alpha}\right)} Q_n(x)\,\frac{Q_k(x)}{\xi_k}\left[a\cos\sqrt{x}t + b\frac{\sin\sqrt{x}t}{\sqrt{x}}\right]d\omega(x), \tag{98}$$

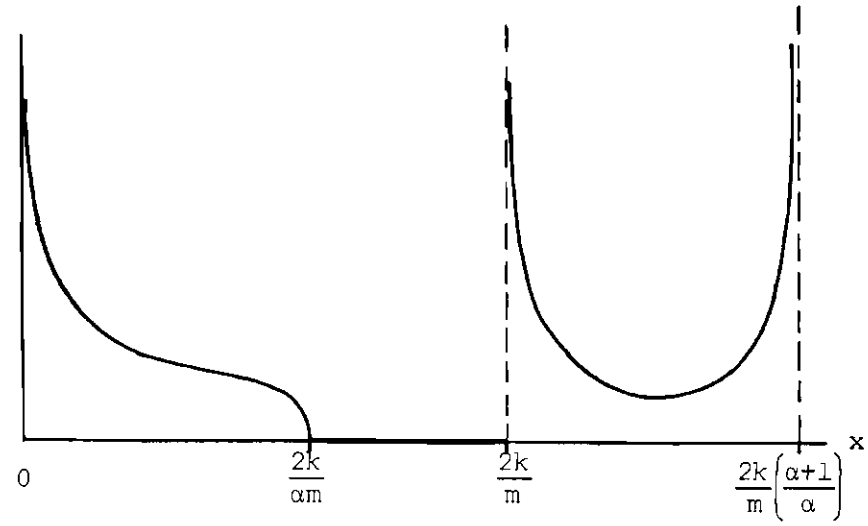
$$-\infty<n<\infty.$$

Note that the symmetric part of the solution,

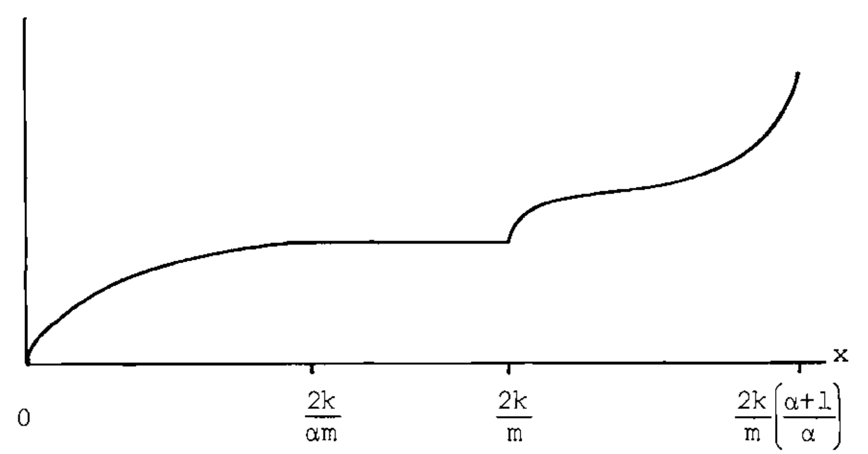
$$w_n(t) = \frac{(1+\delta_{0k})}{2}\int_0^{2\frac{k}{m}\left(\frac{\alpha+1}{\alpha}\right)} R_n(x)\,\frac{R_k(x)}{\gamma_k}\left[a\cos\sqrt{x}t + b\frac{\sin\sqrt{x}t}{\sqrt{x}}\right]d\alpha(x),$$

can in itself be interpreted as the solution of the half-infinite diatomic chain with an initial half-mass and free end (Figure 9b). Similarly the antisymmetric part of the solution,

$$z_n(t) = \frac{(1-\delta_{0k})}{2}\int_0^{2\frac{k}{m}\left(\frac{\alpha+1}{\alpha}\right)} Q_n(x)\,\frac{Q_k(x)}{\xi_k}\left[a\cos\sqrt{x}t + b\frac{\sin\sqrt{x}t}{\sqrt{x}}\right]d\omega(x),$$



Weight Function



Integrator

Figure 9d. Weight Function and Integrator for Symmetric Mode of an Infinite Diatomic Chain

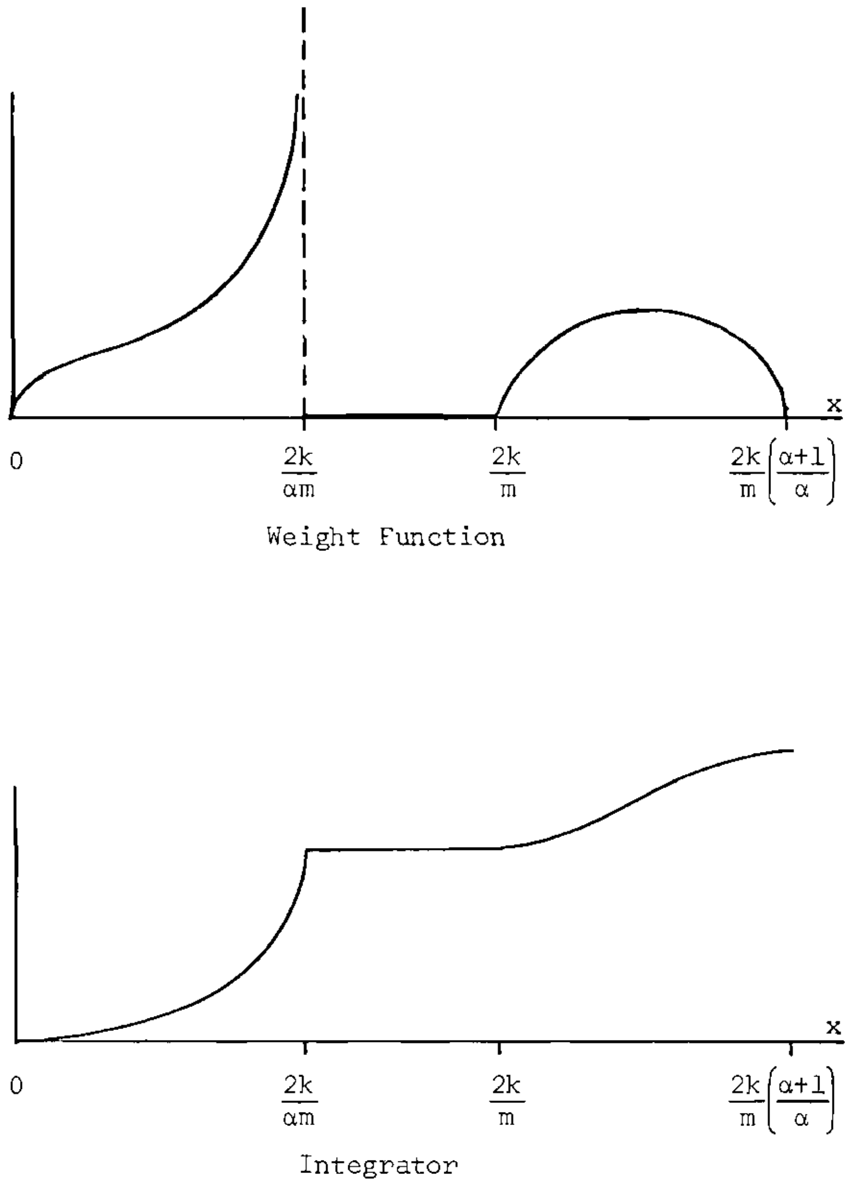


Figure 9e. Weight Function and Integrator for the Antisymmetric Mode of an Infinite Diatomic Chain

yields a solution of the half-infinite diatomic chain with constrained left end (Figure 9c).

It is well known that finite truncations (consisting of 2N+1 masses symmetrically located about the middle mass) of the infinite system have solutions which are superpositions of (2N+1) normal modes. The squares of the natural frequencies of these normal modes are precisely the zeros of the polynomials $R_{N+1}(x)$ and $Q_{N+1}(x)$, and these zeros lie in those subintervals of the interval of orthogonality on which the weight function is not zero. This observation illustrates the fact that the qualitative properties of the weight functions in the symmetric and antisymmetric parts of the solution (98) are determined by the frequency spectrum of the infinite diatomic chain. For convenient reference these weight functions and the corresponding integrators are displayed graphically (see Figures 9d and 9e).

An Infinite Uniform Chain with a Single Isotopic Impurity

Consider the infinite uniform chain of masses m with an isotopic impurity βm in the middle position and each mass linearly coupled to each of its nearest neighbors (see Figure 10a).

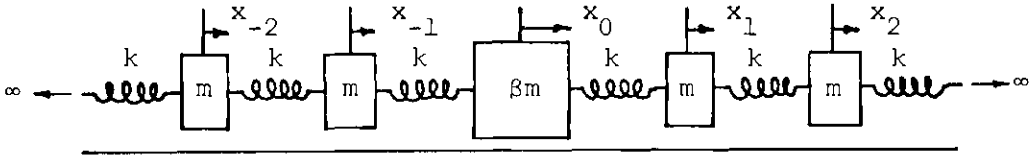


Figure 10a. An Infinite Uniform Chain with a Single Isotopic Impurity

The corresponding symmetric part of this infinite chain is the half-infinite uniform chain with initial impurity and free end shown in Figure 10b.

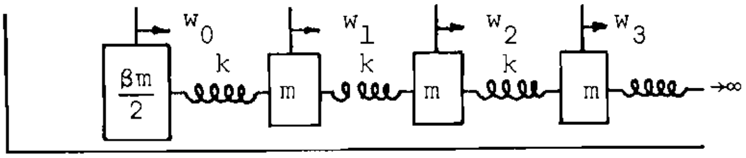


Figure 10b. A Half-Infinite Uniform Chain with Initial Isotopic Impurity

The corresponding polynomials are easily found to be $R_n(x) = M_n^{(1,\beta)}\left(\frac{m}{k}x\right)$; hence from (81)

$$R_n(x) = \frac{(-1)^n 4^n (n!)^2}{(2n)!}\left[\frac{\beta}{2}P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{m}{2k}x-1\right) - \left(\frac{2-\beta}{2}\right)\left(\frac{2n-1}{2}\right)P_{n-1}^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{m}{2k}x-1\right)\right],$$

$$n\geq 1,$$

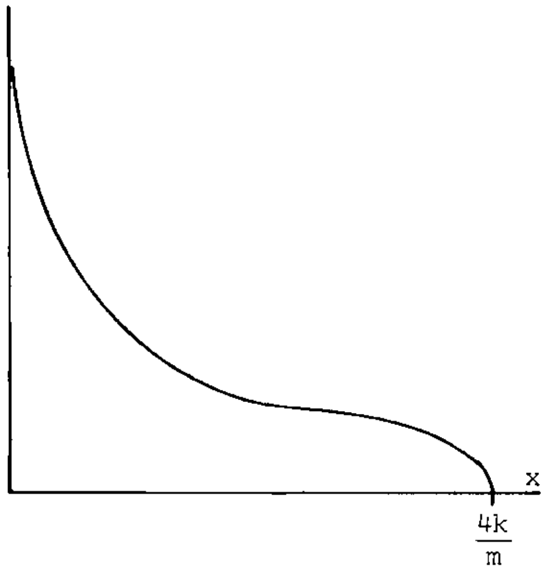
and the corresponding normalizing factors are

$$\gamma_0 = 1, \quad \gamma_n = \frac{\beta}{2}, \quad n\geq 1.$$

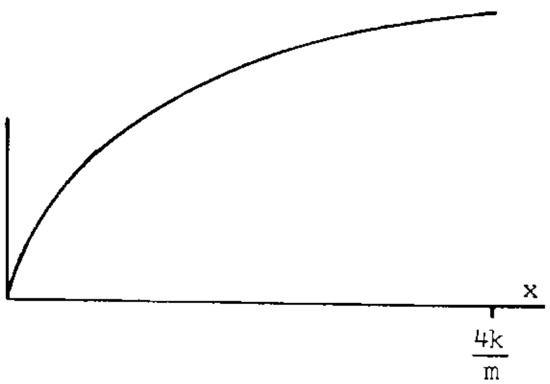
For $\beta>1$ Lemma 12 yields the weight function

$$\rho(x) = \frac{d\alpha(x)}{dx} = \frac{\beta m}{\pi}\sqrt{\frac{4k-mx}{mx}}\,\frac{1}{4k-\beta(2-\beta)mx}, \quad 0\leq x\leq \frac{4k}{m},$$

which is sketched in Figure 10c.



Weight Function ($\beta>1$)



Integrator ($\beta>1$)

Figure 10c. Weight Function and Integrator for the Symmetric Mode of an Infinite Uniform Chain with a Single Isotopic Impurity

For $\beta<1$ the weight function is a generalized function. Lemma 12 shows that the corresponding integrator is given by

$$\alpha(x) = \begin{cases} \int_0^x \rho(t)dt, & 0 \le x < \frac{4k}{m\beta(2-\beta)} , \\ \\ \int_0^{\frac{4k}{m\beta(2-\beta)}} \rho(t)dt + \frac{2(1-\beta)}{(2-\beta)} , & x = \frac{4k}{m\beta(2-\beta)} , \end{cases}$$

where

$$\rho(t) = \begin{cases} \frac{\beta m}{\pi} \sqrt{\frac{4k-mx}{mx}} \frac{1}{4k - \beta(2-\beta)mx} , & 0 < x \le \frac{4k}{m} , \\ \\ 0, & \frac{4m}{k} < x \le \frac{4k}{m\beta(2-\beta)}. \end{cases}$$

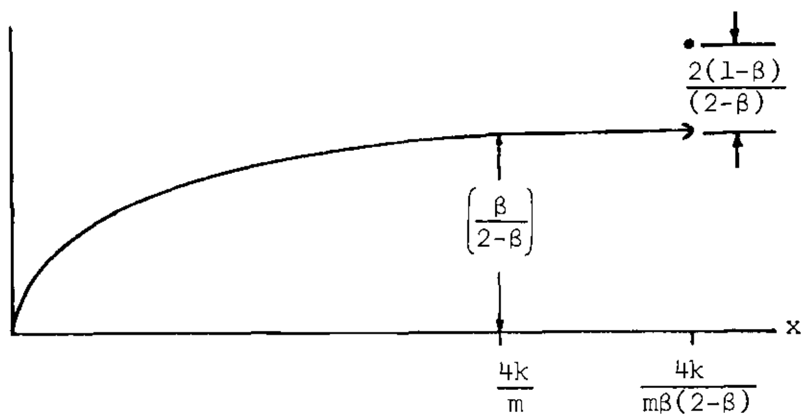


Figure 10d. Integrator for the Symmetric Mode of an Infinite Uniform Chain with a Single Isotopic Impurity ($\beta<1$)

The antisymmetric part of this infinite chain is the half-infinite uniform chain with end constrained (see Figure 10e).

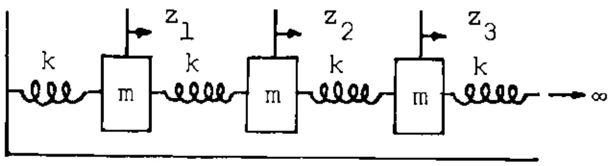


Figure 10e. A Half-Infinite Uniform Chain

With the use of [11] it is easily shown that the polynomials Q_n are classical, in fact

$$Q_n(x) = (-1)^{n-1} \frac{4^n(n!)^2}{2(2n)!} P_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{m}{2k}x-1\right), \quad n\geq 1.$$

The Q_n ($n\geq 1$) are orthogonal on $\left[0, \frac{4k}{m}\right]$ with respect to the weight function

$$\rho(x) = \frac{d\omega(x)}{dx} = \frac{m}{2\pi k^2}\sqrt{4kmx-m^2x^2}$$

and the normalizing factors are $\xi_n=1$, $n\geq 0$.

An Infinite Diatomic Chain with a Heavy Middle Mass

This example presents the constituents needed for the solution of the infinite diatomic chain of alternate masses m and αm in which one mass originally m has been replaced by 2m (see Figure 11a).

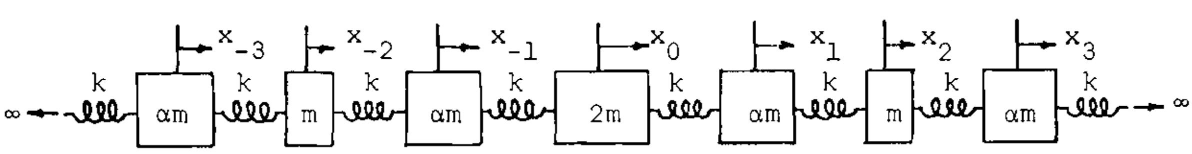


Figure 11a. An Infinite Diatomic Chain with a Heavy Middle Mass

The corresponding symmetric part of this system is the half-infinite diatomic chain with free end (see Figure 11b).

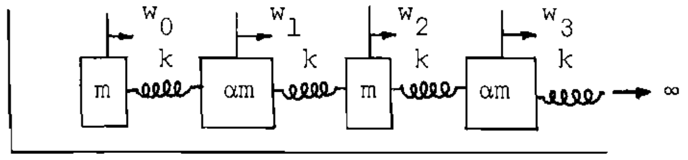


Figure 11b. A Half-Infinite Diatomic Chain with Free End

The symmetric mode polynomials are given by $R_n(x) = M_n^{(\alpha,2)}\left(\frac{m}{k}x\right)$; hence the use of (80) and (78) shows that

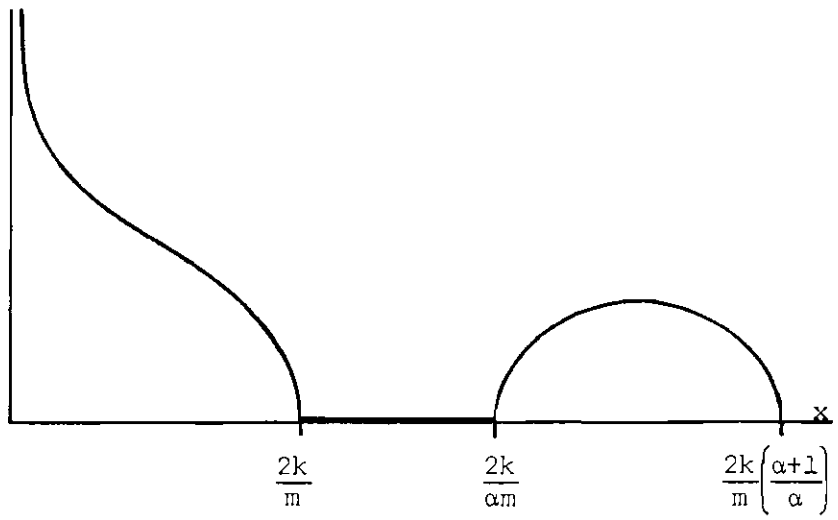
$$R_0 = 1,\; R_{2n}(x) = \frac{4^n(n!)^2}{(2n)!}\left[P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{\alpha m^2x^2}{2k^2} - \frac{mx}{k}(\alpha+1) + 1\right)\right.$$

$$\left. + \left(\frac{\alpha m}{2k}x-1\right)P_{n-1}^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{\alpha m^2x^2}{2k^2} - \frac{mx}{k}(\alpha+1) + 1\right)\right], \quad n\geq 1,$$

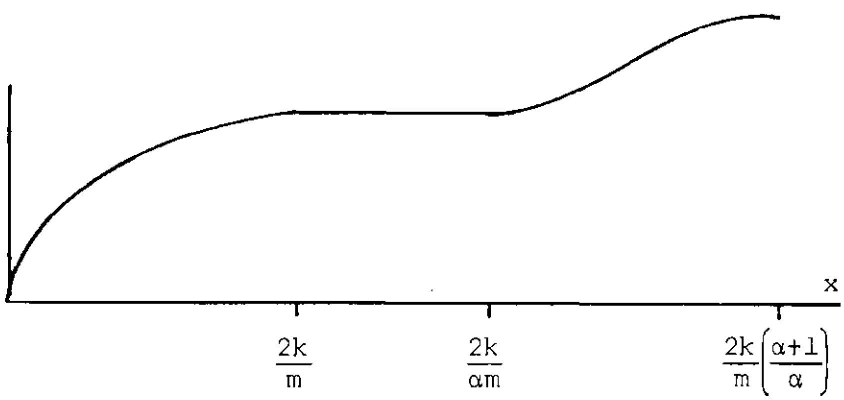
$$R_{2n+1}(x) = -\frac{4^n(n!)^2}{(2n)!}\left[P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\frac{\alpha m^2x^2}{2k^2} - \frac{mx}{k}(\alpha+1) + 1\right)\right.$$

$$\left. + \left(\frac{mx}{2k} - 1\right)\frac{2n+2}{2n+1}P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{\alpha m^2x^2}{2k^2} - \frac{mx}{k}(\alpha+1) + 1\right)\right], \quad n\geq 0.$$

For $\alpha<1$ the weight function (88) is given by (see Figure 11c)



Weight Function (α<1)



Integrator (α<1)

Figure 11c. Weight Function and Integrator for the Symmetric Mode of an Infinite Diatomic Chain with a Heavy Middle Mass

$$\rho(x) = \frac{d\alpha(x)}{dx} = \begin{cases} \dfrac{m}{2\pi k} \dfrac{\sqrt{\left(\frac{2k}{\alpha m} - x\right)\left(\frac{2k}{m} - x\right)\left(\frac{2k}{m}\left(\frac{1+\alpha}{\alpha}\right) - x\right)}}{\sqrt{x}\ \left|\frac{k}{m}\left(\frac{\alpha+1}{\alpha}\right) - x\right|}, & 0<x<\frac{2k}{m},\ \frac{2k}{\alpha m}<x<\frac{2k}{m}\left(\frac{1+\alpha}{\alpha}\right) \\ 0 & , \frac{2k}{m}\le x\le \frac{2k}{\alpha m}, \end{cases}$$

and the normalizing factors are

$$\gamma_{2n} = 1\ (n\geq 0),\quad \gamma_{2n+1} = \frac{1}{\alpha}\ (n\geq 0).$$

For $\alpha>1$ the weight function is a generalized function and the corresponding integrator (see Figure 11d), given by Lemma 15, is

$$\alpha(x) = \begin{cases} \int_0^x \rho(t)dt, & 0\le x<\frac{k}{m}\left(\frac{\alpha+1}{\alpha}\right), \\ \int_0^x \rho(t)dt + \left(\frac{\alpha-1}{\alpha}\right), & \frac{k}{m}\left(\frac{\alpha+1}{\alpha}\right)\le x\le \frac{2k}{m}\left(\frac{\alpha+1}{\alpha}\right). \end{cases}$$

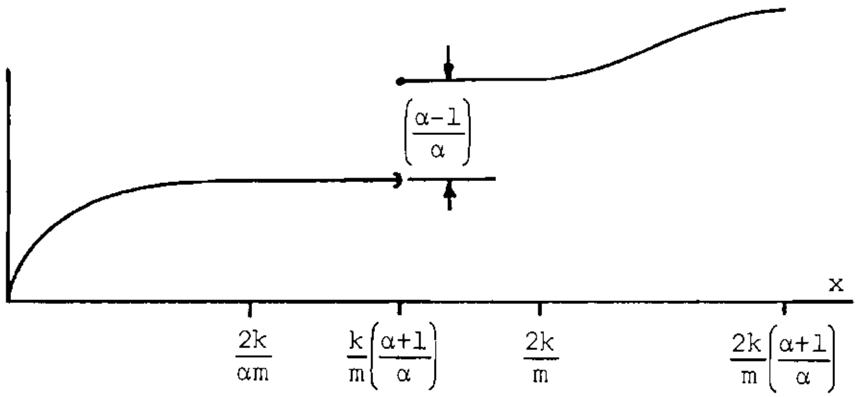


Figure 11d. The Integrator for the Symmetric Mode of an Infinite Diatomic Chain with a Heavy Middle Mass ($\alpha>1$)

The antisymmetric mode of this infinite chain is precisely the same as the antisymmetric mode of the infinite diatomic chain of the first example.

By using the procedures developed in Chapter III, the solutions for the equations of motion of planar arrays with isotopic impurities may be given. The presentation of such an example is omitted here because of its excessive length.

APPENDIX A

THE QUADRATURE FORMULA FOR DOUBLE INTEGRALS

This appendix contains a development of the quadrature formula used in Chapter III to obtain solutions of finite truncations of infinite two-dimensional initial-value problems. Several lemmas are first proved and the main result, the quadrature formula for double integrals, is stated as Theorem A.1.

Lemma A.1. Let N be a positive integer and f a function of one real variable defined on [a,b] and having a continuous derivative of order 2N on (a,b). Let $x_1 < x_2 < \ldots < x_N$ be N points of [a,b]. Then there exists a unique polynomial H_{2N-1} of degree $\leq (2N-1)$ such that

$$H_{2N-1}(x_r) = f(x_r),\ H'_{2N-1}(x_r) = f'(x_r),\quad r=1,2,\ldots,N.$$

Furthermore for every $x \in [a,b]$,

$$f(x) = H_{2N-1}(x) + \frac{f^{(2N)}(\xi)}{(2N)!} \prod_{r=1}^{N} (x-x_r)^2 \tag{A.1}$$

for some $\xi \in (a,b)$.

Proof. For each i=1,2,...,N, define the polynomial $\ell_i(x) = \prod_{\substack{p=1\\p\neq i}}^{N} \left(\frac{x-x_p}{x_i-x_p}\right)$.

It is easily seen that degree $\ell_i = (N-1)$, and $\ell_i(x_r) = \delta_{ir}$ for each

$i=1,2,\ldots,N$, $r=1,2,\ldots,N$. Let H_{2N-1} be the polynomial defined by

$$H_{2N-1}(x) = \sum_{i=1}^{N} [\ell_i(x)]^2 \{f(x_i) + (x-x_i)[f'(x_i) - 2\ell_i'(x_i)f(x_i)]\}.$$

Clearly degree $H_{2N-1} \le (2N-1)$, and a straightforward calculation shows that

$$H_{2N-1}(x_r) = \sum_{i=1}^{N} \delta_{ir}\{f(x_i) + (x_r-x_i)[f'(x_i) - 2\ell_i'(x_i)f(x_i)] = f(x_r)$$

and

$$\frac{d}{dx} H_{2N-1}(x_r) = \sum_{i=1}^{N} 2\delta_{ir}\ell_i'(x_r)\{f(x_i) + (x_r-x_i)[f'(x_i) - \ell_i'(x_i)f(x_i)]\}$$

$$+ \sum \delta_{ir}[f'(x_i) - 2\ell_i'(x_i)f(x_i)] = f'(x_r).$$

Thus existence is established. To show uniqueness, suppose P is a polynomial of degree $\le(2N-1)$ satisfying

$$P(x_r) = f(x_r), \quad P'(x_r) = f'(x_r), \quad r=1,2,\ldots,N.$$

Then $D(x) = H_{2N-1}(x) - P(x)$ is either the identically zero polynomial or a polynomial of degree $\le(2N-1)$ with $D(x_r) = 0$, $D'(x_r) = 0$ for $r=1,2,\ldots,N$. If $D(x) \equiv 0$, the uniqueness of H_{2N-1} is established. Otherwise, by Rolle's theorem, the polynomial D' has a zero ξ_i on each of the open intervals (x_i,x_{i+1}), $i=1,2,\ldots,N$. Thus D' is a polynomial of degree $\leq(2N-2)$ and has $(2N-1)$ distinct zeros at $x_1<\xi_1<x_2<\xi_2<\ldots<x_{N-1}<\xi_{N-1}<x_N$. It follows that $D(x) \equiv 0$, and hence H_{2N-1} is unique. To establish (A.1), for $x\epsilon[a,b]$ and $x\neq x_r$, $r=1,2,\ldots,N$, define $R(x) = f(x) - H_{2N-1}(x)$; and let ψ be defined on $[a,b]$ by

$$\psi(t) = f(t) - H_{2N-1}(t) - R(x) \prod_{r=1}^{N} \left(\frac{t-x_r}{x-x_r}\right)^2 .$$

Then

$$\psi(x_r) = f(x_r) - H_{2N-1}(x_r) = 0, \quad r=1,2,\ldots,N,$$

and

$$\psi(x) = f(x) - H_{2N-1}(x) - R(x) = 0.$$

Hence ψ has $(N+1)$ zeros at $x,x_1,x_2,\ldots,x_N$. By Rolle's theorem, ψ' has a zero at each of N points $\xi_1,\xi_2,\ldots,\xi_N$ each of which is distinct from $x_1,x_2,\ldots,x_N$. A direct calculation shows that

$$\psi'(x_r) = f'(x_r) - H'_{2N-1}(x_r) = 0, \quad \text{for } r=1,2,\ldots,N \left({}'\sim\frac{d}{dt}\right).$$

Thus ψ' has 2N distinct zeros at $\xi_1,\xi_2,\ldots,\xi_N$ and $x_1,x_2,\ldots,x_N$. It is easily shown that $\psi^{(2N)}(\xi) = 0$ for some ξ in the open interval (a,b). But

$$\psi^{(2N)}(\xi) = f^{(2N)}(\xi) - \frac{R(x)(2N)!}{\prod_{r=1}^{N} (x-x_r)^2} .$$

Consequently

$$R(x) = \frac{f^{(2N)}(\xi)}{(2N)!} \prod_{r=1}^{N} (x-x_r)^2 \quad \text{and} \quad f(x) = H_{2N-1}(x) + \frac{f^{(2N)}(\xi)}{(2N)!} \prod_{r=1}^{N} (x-x_r)^2.$$

This completes the proof.

Definition A.1. Let $\{\phi_n\}$ be a sequence of monic polynomials orthogonal on an interval $[a,b]$ with respect to a normalized integrator α. For $N \geq 1$, let $x_1, x_2, \ldots, x_N$ be the N distinct zeros of ϕ_N. The *Christoffel numbers* associated with the zeros of ϕ_N are given by

$$\lambda_i = \int_a^b \frac{\phi_N(x)}{(x-x_i)\phi_N'(x_i)} \, d\alpha(x) \quad \text{for } i=1,2,\ldots,N.$$

Several equivalent definitions of Christoffel numbers are given in the following lemma. For a proof of their equivalence see [23, p.48].

Lemma A.2. Let α be a normalized integrator on $[a,b]$ and let $\{\phi_n\}$ $(n \geq 0)$ be the sequence of monic polynomials orthogonal on $[a,b]$ with respect to α which satisfy the three-term recurrence

$$\phi_0 = 1$$

$$\phi_1(x) = x + b_0$$

$$\phi_{n+1}(x) = (x+b_n)\phi_n(x) - c_n\phi_{n-1}(x), \quad n \geq 1.$$

For each $N \geq 1$, let $x_1, x_2, \ldots, x_N$ denote the N real zeros of ϕ_N. Then the following definitions are equivalent to Definition A.1.

$$\text{(i)}\quad \lambda_i = \int_a^b \left[\frac{\phi_N(x)}{(x-x_i)\phi_N'(x_i)}\right]^2 d\alpha(x), \quad i=1,2,\ldots,N$$

$$\text{(ii)}\quad \lambda_i = -\frac{c_0c_1\cdots c_N}{\phi_{N+1}(x_i)\phi_N'(x_i)}, \quad i=1,2,\ldots,N,\ c_0=1$$

$$\text{(iii)}\quad \lambda_i = \frac{c_0c_1\cdots c_{N-1}}{\phi_{N-1}(x_i)\phi_N'(x_i)}, \quad i=1,2,\ldots,N,\ c_0=1$$

Lemma A.3. Let α be a normalized integrator on $[a,b]$ and $\{\phi_n\}$ $(n\geq 0)$ the monic polynomials which are orthogonal on $[a,b]$ with respect to α. For any integer $N\geq 1$, let $x_1,x_2,\ldots,x_N$ denote the zeros of ϕ_N, and let $\lambda_1,\lambda_2,\ldots,\lambda_N$ be the corresponding Christoffel numbers. Then if P is any polynomial of degree $\leq(2N-1)$,

$$\int_a^b P(x)d\alpha(x) = \sum_{i=1}^{N} \lambda_i P(x_i).$$

Proof. By Lemma A.1 it follows that

$$P(x) = \sum_{i=1}^{N} [\ell_i(x)]^2\{P(x_i) + (x-x_i)[P'(x_i) - 2\ell_i'(x_i)P(x_i)]\},$$

where

$$[\ell_i(x)]^2 = \prod_{\substack{p=1\\p\neq i}}^{N} \left(\frac{x-x_p}{x_i-x_p}\right)^2 = \left[\frac{\phi_N(x)}{(x-x_i)\phi_N'(x_i)}\right]^2.$$

Thus

$$\int_a^b P(x)d\alpha(x) = \sum_{i=1}^N P(x_i) \int_a^b [\ell_i(x)]^2 d\alpha(x)$$

$$+ \sum_{i=1}^N [P'(x_i)-2\ell_i'(x_i)P(x_i)] \int_a^b [\ell_i(x)]^2(x-x_i)d\alpha(x).$$

By conclusion (i) of Lemma A.2, $\int_a^b [\ell_i(x)]^2 d\alpha(x) = \lambda_i$. Note that

$$[\ell_i(x)]^2(x-x_i) = \phi_N(x)\left\{\frac{\phi_N(x)}{(x-x_i)[\phi_N'(x_i)]^2}\right\}$$

and

$$\text{degree } \left\{\frac{\phi_N(x)}{(x-x_i)[\phi_N'(x_i)]^2}\right\} = N - 1, \quad \text{for each } i=1,2,\ldots,N.$$

Since ϕ_N is orthogonal to all polynomials of degree $\leq(N-1)$, it follows that $\int_a^b [\ell_i(x)]^2(x-x_i)d\alpha(x) = 0$, for $i=1,2,\ldots,N$. Hence $\int_a^b P(x)d\alpha(x) = \sum_{i=1}^N \lambda_i P(x_i)$. This completes the proof.

Lemma A.4. Let α be a normalized integrator and $\{\phi_n\}$ the monic polynomials which are orthogonal on $[a,b]$ with respect to α. For any positive integer N let $\lambda_1,\lambda_2,\ldots,\lambda_N$ denote the Christoffel numbers associated with the N zeros of ϕ_N. Then $\sum_{i=1}^N \lambda_i = 1$.

Proof. Apply Lemma A.3 to $P(x) = 1$ on $[a,b]$. Then since α is normalized,

$$1 = \int_a^b 1 d\alpha(x) = \sum_{i=1}^N \lambda_i .$$

The preceding lemmas may be applied to obtain the quadrature formula for double integrals over a rectangle $R = [a,b] \times [c,d]$.

<u>Theorem A.1</u>. Let α be a normalized integrator on $[a,b]$ and $\{\phi_n\}$ $(n \geq 0)$ the associated monic polynomials orthogonal on $[a,b]$ with respect to α. Let ω be a normalized integrator on $[c,d]$ and $\{\psi_n\}$ $(n \geq 0)$ the associated monic polynomials orthogonal on $[c,d]$ with respect to ω. For each positive integer N let $x_1, x_2, \ldots, x_N$ denote the N zeros of ϕ_N and $\lambda_1, \lambda_2, \ldots, \lambda_N$ the associated Christoffel numbers. For each positive integer M let $y_1, y_2, \ldots, y_M$ denote the M zeros of ψ_M and $\kappa_1, \kappa_2, \ldots, \kappa_M$ the associated Christoffel numbers. Let $\tau_N = \int_a^b \phi_N^2(x)\,d\alpha(x)$ and $\sigma_M = \int_c^d \psi_M^2(y)\,d\omega(y)$. Let f be defined on $R = [a,b] \times [c,d]$, and suppose that $\partial^{2N} f/\partial x^{2N}$ and $\partial^{2M} f/\partial y^{2M}$ are continuous on R.

Then

$$\int_a^b \int_c^d f(x,y)\,d\omega(y)\,d\alpha(x) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_i \kappa_j f(x_i, y_j)$$

$$+ \frac{1}{(2N)!} \frac{\partial^{2N} f}{\partial x^{2N}} (\xi_1, \eta_1) \tau_N + \frac{1}{(2M)!} \frac{\partial^{2M} f}{\partial y^{2M}} (\xi_2, \eta_2) \sigma_M$$

for some $\xi_1, \xi_2 \epsilon [a,b]$ and some $\eta_1, \eta_2 \epsilon [c,d]$.

Proof. For any fixed $x \epsilon [a,b]$ Lemma A.1 may be applied to $f(x, \cdot)$ to obtain

$$f(x,y) = H_{2M-1}(x,y) + \frac{1}{(2M)!} \frac{\partial^{2M} f}{\partial y^{2M}} (x,\eta) \prod_{s=1}^{M} (y - y_s)^2,$$

where $H_{2M-1}(x,\cdot)$ is a polynomial of degree $\leq(2M-1)$,

$$H_{2M-1}(x,y_j) = f(x,y_j) \text{ for } j=1,2,\ldots,N, \text{ and } \eta\epsilon(c,d).$$

It follows from Lemma A.3 that

$$\int_c^d f(x,y)d\omega(y) = \sum_{j=1}^{M} \kappa_j f(x,y_j) + \frac{1}{(2M)!}\int_c^d \frac{\partial^{2M}f}{\partial y^{2M}}(x,\eta)\psi_M^2(y)d\omega(y)$$

and hence that

$$\int_a^b \int_c^d f(x,y)d\omega(y)d\alpha(x) = \sum_{j=1}^{M} \kappa_j \int_a^b f(x,y_j)d\alpha(x)$$

$$+ \frac{1}{(2M)!}\int_a^b \int_c^d \frac{\partial^{2M}f}{\partial y^{2M}}(x,\eta)\psi_M^2(y)d\omega(y)d\alpha(x).$$

For each $j=1,2,\ldots,M$, apply Lemma A.1 to $f(\cdot,y_j)$ to obtain

$$f(x,y_j) = H_{2N-1}(x,y_j) + \frac{1}{(2N)!}\frac{\partial^{2N}f}{\partial x^{2N}}(\xi_j,y_j)\prod_{r=1}^{N}(x-x_r)^2,$$

where $H_{2N-1}(\cdot,y_j)$ is a polynomial of degree $\leq(2N-1)$,

$$H_{2N-1}(x_i,y_j) = f(x_i,y_j) \text{ for } i=1,2,\ldots,N, \text{ and } \xi_j\epsilon(a,b).$$

It follows by Lemma A.3 that

$$\int_a^b f(x,y_j)d\alpha(x) = \sum_{i=1}^{N} \lambda_i f(x_i,y_j) + \frac{1}{(2N)!}\int_a^b \frac{\partial^{2N}f}{\partial x^{2N}}(\xi_j,y_j)\phi_N^2(x)d\alpha(x)$$

and hence that

$$\int_a^b \int_c^d f(x,y)d\omega(y)d\alpha(x) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_i \kappa_j f(x_i,y_j)$$

$$+ \frac{1}{(2N)!} \sum_{j=1}^{M} \kappa_j \int_a^b \frac{\partial^{2N} f}{\partial x^{2N}} (\xi_j,y_j)\phi_N^2(x)d\omega(y)d\alpha(x)$$

$$+ \frac{1}{(2M)!} \int_a^b \int_c^d \frac{\partial^{2M} f}{\partial y^{2M}} (x,\eta)\psi_M^2(y)d\omega(y)d\alpha(x).$$

Let U_M and L_M denote, respectively, the maximum and minimum values of $\frac{\partial^{2M} f}{\partial y^{2M}}$ on R.

Then

$$L_M\sigma_M = L_M \int_a^b \int_c^d \psi_M^2(y)d\omega(y)d\alpha(x) \leq \int_a^b \int_c^d \frac{\partial^{2M} f}{\partial y^{2M}} (x,\eta)\psi_M^2(y)d\omega(y)d\alpha(x)$$

$$\leq U_M \int_a^b \int_c^d \psi_M^2(y)d\omega(y)d\alpha(x) = U_M\sigma_M.$$

Since $\sigma_M \frac{\partial^{2M} f}{\partial y^{2M}}$ is continuous on R, it assumes every value between its maximum and minimum values; in particular

$$\sigma_M \frac{\partial^{2M} f}{\partial y^{2M}} (\xi_2,\eta_2) = \int_a^b \int_c^d \frac{\partial^{2M} f}{\partial y^{2M}} (x,\eta)\psi_M^2(y)d\omega(y)d\alpha(x)$$

for some $\xi_2\epsilon[a,b]$ and some $\eta_2\epsilon[c,d]$. Let U_N and L_N denote, respectively, the maximum and minimum values of $\frac{\partial^{2N} f}{\partial x^{2N}}$ on R.

Then

$$\sum_{j=1}^{M}\kappa_j L_N \tau_N = \sum_{j=1}^{M}\kappa_j L_N \int_a^b \phi_N^2(x)d\alpha(x) \le \sum_{j=1}^{M}\mu_j \int_a^b \frac{\partial^{2N}f}{\partial x^{2N}}(\xi_j,y_j)\phi_N^2(x)d\alpha(x)$$

$$\le \sum_{j=1}^{M} \kappa_j U_N \int_a^b \phi_N^2(x)d\alpha(x) = \sum_{j=1}^{M} \kappa_j U_N \tau_N.$$

But by Lemma 4, $\sum_{j=1}^{M} \kappa_j = 1$; hence

$$L_N\tau_N \le \sum_{j=1}^{M} \kappa_j \int_a^b \frac{\partial^{2N}f}{\partial x^{2N}}(\xi_j,y_j)\phi_N^2(x)d\alpha(x) \le U_N\tau_N.$$

By continuity of $\tau_N \frac{\partial^{2N}f}{\partial x^{2N}}$ on R, there exist $\xi_1\epsilon[a,b]$ and $\eta_1\epsilon[c,d]$ such that

$$\tau_N \frac{\partial^{2N}f}{\partial x^{2N}}(\xi_1,\eta_1) = \sum_{j=1}^{M} \kappa_j \int_a^b \frac{\partial^{2N}f}{\partial x^{2N}}(\xi_j,y_j)\phi_N^2(x)d\alpha(x).$$

Consequently

$$\int_a^b \int_c^d f(x,y)d\omega(y)d\alpha(x) = \sum_{i=1}^{N}\sum_{j=1}^{M} \lambda_i\kappa_j f(x_i,y_j)$$

$$+ \frac{\tau_N}{(2N)!}\frac{\partial^{2N}f}{\partial x^{2N}}(\xi_1,\eta_1) + \frac{\sigma_M}{(2M)!}\frac{\partial^{2M}f}{\partial y^{2M}}(\xi_2,\eta_2),$$

which completes the proof.

APPENDIX B

TABULATION OF SOLUTIONS

For convenient reference the following pages contain a tabulation of physical parameters and the required polynomials, intervals of orthogonality, and weight functions for solutions of various physical systems of the type treated in Chapters II and III. The following notation is used throughout this section:

[i] $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials described in Szegö [23];

[ii] $L_n^{(\alpha)}$ are the Laguerre polynomials described in Szegö [23];

[iii] $M_n^{(\alpha,\beta)}$ are the non-classical polynomials described in Chapter IV.

Table 1a. Symmetric Solution for Infinite Chains

	m_n	k_n	γ_n	$R_n(n\geq 0)$	$[a,b]$	$\frac{d\alpha(x)}{dx}$
1.	$m_n=m,\ n\geq 0$	$k_n=k,\ n\geq 1$	$\gamma_0=1$ $\gamma_n=\frac{1}{2},\ n\geq 1$	$r_n P_n^{(-1/2,-1/2)}\left(\frac{mx}{2k}-1\right)$ $r_n=\frac{(-1)^n 4^n [n!]^2}{[(2n)!]}$	$\left[0,\frac{4k}{m}\right]$	$\frac{m}{\pi\sqrt{4kmx-m^2x^2}}$
2.	$m_0=2m$ $m_n=m,\ n\geq 0$	$k_n=k,\ n\geq 1$	1	$r_n P_n^{(1/2,-1/2)}\left(\frac{mx}{2k}-1\right)$ $r_n=\frac{(-1)^n 4^n [n!]^2}{[(2n)!]}$	$\left[0,\frac{4k}{m}\right]$	$\frac{m}{2\pi k}\sqrt{\frac{4k-mx}{mx}}$
3.	$m_0=2m$ $m_n=(n+1)^2 m,\ n\geq 1$	$k_n=n(n+1)k,\ n\geq 1$	$\frac{1}{(n+1)^2}$	$r_n P_n^{(1/2,1/2)}\left(\frac{mx}{2k}-1\right)$ $r_n=\frac{(-1)^n 4^n [n!]}{[(2n+1)!]}$	$\left[0,\frac{4k}{m}\right]$	$\frac{m}{2\pi k^2}\sqrt{4kmx-m^2x^2}$
4.	$m_0=2m$ $m_n=(2n+1)^2 m,\ n\geq 1$	$k_n=(4n-1)^2,\ n\geq 1$	$\frac{1}{(2n+1)^2}$	$r_n P_n^{(-1/2,1/2)}\left(\frac{mx}{2k}-1\right)$ $r_n=\frac{(-1)^n 4^n [n!]}{[(2n+1)!]}$	$\left[0,\frac{4k}{m}\right]$	$\frac{m}{2\pi k}\sqrt{\frac{mx}{4k-mx}}$
5.	$m_0=\beta m,\ (\beta\geq 1)$ $m_n=m,\ n\geq 1$	$k_n=k,\ n\geq 1$	$\frac{\beta}{2}$	$M_n^{(1,\beta)}\left(\frac{mx}{2k}-1\right)$	$\left[0,\frac{4k}{m}\right]$	$\frac{\beta m}{\pi}\sqrt{\frac{4k-mx}{mx}}\ \frac{1}{4k-\beta(2-\beta)mx}$
6.	$m_0=\beta m,(\beta<1)$ $m_n=m,\ n\geq 1$	$k_n=k,\ n\geq 1$	$\frac{\beta}{2}$	$M_n^{(1,\beta)}\left(\frac{mx}{2k}-1\right)$	$\left[0,\frac{4k}{m\beta(2-\beta)}\right]$	$\frac{\beta m}{\pi}\sqrt{\frac{4k-mx}{mx}}\ \frac{1}{4k-\beta(2-\beta)mx}+\frac{2(1-\beta)}{(2-\beta)}\delta\left(x-\frac{4k}{m\beta(2-\beta)}\right)$

$A_n=-\frac{m_n}{k_{n+1}}\ (n\geq 0),\quad B_n=1+\frac{k_n}{k_{n+1}}\ (n\geq 0),\quad C_n=\frac{k_n}{k_{n+1}}\ (n\geq 1).$

Table 1b. Antisymmetric Solution for Infinite Chains

	m_n	k_n	ξ_n	Q_n $(n\geq 1)$	$[c,d]$	$\frac{d\omega(x)}{dx}$
1.	$m_n=m,\ n\geq 0$	$k_n=k,\ n\geq 1$	1	$q_n P_{n-1}^{(1/2,1/2)}\left(\frac{mx}{2k}-1\right)$ $q_n=\frac{(-1)^{n-1}4^n[n!]^2}{2[(2n)!]}$	$\left[0,\ \frac{4k}{m}\right]$	$\frac{m}{2\pi k^2}\sqrt{4kmx-m^2x^2}$
2.	$m_0=2m$ $m_n=m,\ n\geq 1$	$k_n=k,\ n\geq 1$	1	$q_n P_{n-1}^{(1/2,1/2)}\left(\frac{mx}{2k}-1\right)$ $q_n=\frac{(-1)^{n-1}4^n[n!]^2}{2[(2n)!]}$	$\left[0,\ \frac{4k}{m}\right]$	$\frac{m}{2\pi k^2}\sqrt{4kmx-m^2x^2}$
3.	$m_0=2m$ $m_n=(n+1)^2m,\ n\geq 1$	$k_n=n(n+1)k,\ n\geq 1$	$\frac{4}{(n+1)^2}$	$q_n P_{n-1}^{(1/2,1/2)}\left(\frac{mx}{2k}-1\right)$ $q_n=\frac{(-1)^{n-1}4^n[n!]^2}{(n+1)[(2n)!]}$	$\left[0,\ \frac{4k}{m}\right]$	$\frac{m}{2\pi k^2}\sqrt{4kmx-m^2x^2}$
4.	$m_0=2m$ $m_n=(2n+1)^2m,\ n\geq 1$	$k_n=(4n^2-1)k,\ n\geq 1$	$\frac{9}{(2n+1)^2}$	$q_n P_{n-1}^{(1/2,1/2)}\left(\frac{mx}{2k}-1\right)$ $q_n=\frac{(-1)^{n-1}3\cdot 4^n[n!]^2}{2[(2n+1)!]}$	$\left[0,\ \frac{4k}{m}\right]$	$\frac{m}{2\pi k^2}\sqrt{4kmx-m^2x^2}$
5.	$m_0=\beta m,\ (\beta\geq 1)$ $m_n=m,\ n\geq 1$	$k_n=k,\ n\geq 1$	1	$q_n P_{n-1}^{(1/2,1/2)}\left(\frac{mx}{2k}-1\right)$ $q_n=\frac{(-1)^{n-1}4^n[n!]^2}{2[(2n)!]}$	$\left[0,\ \frac{4k}{m}\right]$	$\frac{m}{2\pi k^2}\sqrt{4kmx-m^2x^2}$
6.	$m_0=\beta m,\ (\beta<1)$ $m_n=m,\ n\geq 1$	$k_n=k,\ n\geq 1$	1	$q_n P_{n-1}^{(1/2,1/2)}\left(\frac{mx}{2k}-1\right)$ $q_n=\frac{(-1)^{n-1}4^n[n!]^2}{2[(2n)!]}$	$\left[0,\ \frac{4k}{m}\right]$	$\frac{m}{2\pi k^2}\sqrt{4kmx-m^2x^2}$

$A_n = -\frac{m_n}{k_{n+1}}\ (n\geq 1),\quad B_n = 1+\frac{k_n}{k_{n+1}}\ (n\geq 1),\quad C_n=\frac{k_n}{k_{n+1}}\ (n\geq 2).$

Table 2. Solutions for Quarter-Planar Arrays

	m_{ij}	ρ_{ij}	μ_{ij}	γ_n	$P_n(x)$	$[a,b]$	$\frac{d\alpha(x)}{dx}$	ξ_n	$Q_n(y)$	$[c,d]$	$\frac{d\omega(y)}{dy}$
1.	$\left(\frac{1}{j+1}\right)m$	$\left(\frac{i}{j+1}\right)\rho$	μ	1	$L_n^{(0)}\left(\frac{m}{k}x\right)$	$[0,\infty)$	$\frac{m}{\rho}e^{-\frac{m}{\rho}x}$	n+1	$L_n^{(1)}\left(\frac{m}{\mu}y\right)$	$[0,\infty)$	$y\frac{m^2}{\mu^2}e^{-\frac{m^2}{\mu^2}y}$
2.	$(2i+1)m$	$i\rho$	$(2i+1)j\mu$	$\frac{1}{2n+1}$	$(-1)^n P_n^{(0,0)}\left(\frac{m}{\rho}x-1\right)$	$\left[0,\frac{2\rho}{m}\right]$	$\frac{m}{2\rho}$	1	$L_n^{(0)}\left(\frac{m}{\mu}y\right)$	$[0,\infty)$	$\frac{m}{\mu}e^{-\frac{m}{\mu}y}$
3.	$\left(\frac{2i+1}{j+1}\right)m$	$\left(\frac{i}{j+1}\right)\rho$	$(2i+1)\mu$	$\frac{1}{2n+1}$	$(-1)^n P_n^{(0,0)}\left(\frac{m}{\rho}x-1\right)$	$\left[0,\frac{2\rho}{m}\right]$	$\frac{m}{2\rho}$	n+1	$L_n^{(1)}\left(\frac{m}{\mu}y\right)$	$[0,\infty)$	$y\frac{m^2}{\mu^2}e^{-\frac{m^2}{\mu^2}y}$
4.	m	ρ	μ	1	$p_n P_n^{(1/2,1/2)}\left(\frac{m}{2\rho}x-1\right)$, $p_n=\frac{(-1)^n 4^{n+1}[(n+1)!]^2}{2[(2n+2)!]}$	$\left[0,\frac{4\rho}{m}\right]$	$\frac{m}{2\pi\rho^2}\sqrt{4\rho mx-m^2x^2}$	1	$q_n P_n^{(1/2,1/2)}\left(\frac{m}{2\mu}y-1\right)$ $q_n=\frac{(-1)^n 4^{n+1}[(n+1)!]^2}{2[(2n+2)!]}$	$\left[0,\frac{4\mu}{m}\right]$	$\frac{m}{2\pi\mu^2}\sqrt{4\mu my-m^2y^2}$
5.	m	$(1-\delta_{0i})\rho$	μ	1	$p_n P_n^{(1/2,-1/2)}\left(\frac{m}{2\rho}x-1\right)$, $p_n=\frac{(-1)^n 4^n[n!]^2}{[(2n)!]}$	$\left[0,\frac{4\rho}{m}\right]$	$\frac{m}{2\rho\pi}\sqrt{\frac{4\rho-mx}{mx}}$	1	$q_n P_n^{(1/2,1/2)}\left(\frac{m}{2\mu}y-1\right)$, $q_n=\frac{(-1)^n 4^{n+1}[(n+1)!]^2}{2[(2n+2)!]}$	$\left[0,\frac{4\mu}{m}\right]$	$\frac{m}{2\pi\mu^2}\sqrt{4\mu my-m^2y^2}$
6.	m	$(1-\delta_{0i})\rho$	$(1-\delta_{0j})\mu$	1	$p_n P_n^{(1/2,-1/2)}\left(\frac{m}{2\rho}x-1\right)$, $p_n=\frac{(-1)^n 4^n[n!]^2}{[(2n)!]}$	$\left[0,\frac{4\rho}{m}\right]$	$\frac{m}{2\rho\pi}\sqrt{\frac{4\rho-mx}{mx}}$	1	$q_n P_n^{(1/2,-1/2)}\left(\frac{m}{2\mu}y-1\right)$, $q_n=\frac{(-1)^n 4^n[n!]^2}{[(2n)!]}$	$\left[0,\frac{4\mu}{m}\right]$	$\frac{m}{2\mu\pi}\sqrt{\frac{4\mu-my}{my}}$
7.	$(2i+1)(2j+1)m$	$i(2j+1)\rho$	$j(2i+1)\mu$	$\frac{1}{2n+1}$	$(-1)^n P_n^{(0,0)}\left(\frac{m}{\rho}x-1\right)$	$\left[0,\frac{2\rho}{m}\right]$	$\frac{m}{2\rho}$	$\frac{1}{2n+1}$	$(-1)^n P_n^{(0,0)}\left(\frac{m}{\mu}y-1\right)$	$\left[0,\frac{2\mu}{m}\right]$	$\frac{m}{2\mu}$

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